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CE 130
Section 1
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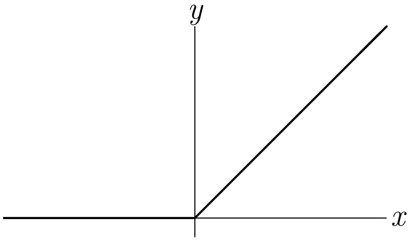
Singularity functions

1. Ramp function

The **Macauley bracket** $<\cdot>$ defines the **ramp function**

$$<x> = \begin{cases} 0, & x \le 0, \\ x, & x > 0. \end{cases}$$



Obviously.

$$<x-a> = \begin{cases} 0, & x \le a, \\ x-a, & x > a, \end{cases}$$

One can further extend the definition of bracket function to higher order power functions

$$<x-a>^n = \begin{cases} 0, & x \le a, \\ (x-a)^n, & x > a \end{cases} \qquad \forall n \ge 0$$

The way to understand the bracket functions is that when the argument inside the bracket is negative (less than zero) the bracket function takes the zero value (*"the show has not started yet "*), and when the argument inside the bracket is positive (greater than zero) the bracket function takes the "face value" of the argument, i.e. a regular function in terms of the argument, e.g.

$$< x-a >^3 = (x-a)^3, \quad \text{if} \quad x-a \ge 0 \quad . \tag{1}$$

One can show that for $n > 0$ the bracket function follows the following rules in differentiation as well as integration

$$\frac{d}{dx}<x-a>^{(n+1)} = (n+1)<x-a>^n,$$

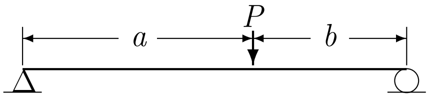


Figure 1: A simply supported beam with a concentrated load

so that

$$\int <x-a>^n \, dx = \frac{1}{n+1} <x-a>^{(n+1)}.$$

This is very similar to the rules in Calculus that we have learned before. However, for definite integrals, there are some differences between the bracket power function and the regular power function. For instance, for $n > 0$,

$$\int_0^y <x-a>^n \, dx = \frac{1}{n+1} <y-a>^{(n+1)}.$$

whereas

$$\int_0^y (x-a)^n dx = \frac{1}{n+1} \Big[(y+1)^{n+1} - (-a)^{n+1} \Big]$$

2. Beam example

To apply bracket function technique to study beam theory, we first look at an example in which a beam is loaded as shown above. The bending moment is given by

$$M(x) = \begin{cases} \dfrac{Pbx}{L}, & x < a \\ \dfrac{Pa(L-x)}{L}, & x > a \end{cases}$$

But

$$\frac{Pa(L-x)}{L} = \frac{Pbx}{L} - P(x-a);$$

therefore the moment can be given by the **single expression**

$$M(x) = \frac{Pbx}{L} - P<x-a>.$$

This can now be integrated twice to give a single expression for the deflection:

$$EIv(x) = \frac{Pbx^3}{6L} - \frac{P}{6} <x-a>^3 + C_1 x + C_2.$$

The boundary condition $v(0) = 0$ yields $C_2 = 0$, while $v(L) = 0$ gives

$$C_1 = -\frac{Pb}{6L}(L^2 - b^2),$$

leading to

$$EIv(x) = \frac{P}{6}\left\{\frac{b}{L}[x^3 - (L^2 - b^2)x] - {<}x - a{>}^3\right\}.$$

3. Heaviside Step function

The derivative of the ramp function is

$$\frac{d}{dx}{<}x{>} = \begin{cases} 0, & x < 0, \\ 1, & x > 0, \end{cases}$$

known as the **Heaviside step function**, in honor of British electrical engineer Oliver Heaviside. There are various notations for it: $H(x)$, $U(x)$, $\mathbf{1}(x)$ and others (the notation ${<}x{>}^0$ is ambiguous because it gives 0^0 for negative x). Let us use $H(x)$, because this is the dominant notation in the scientific literature. Here H is in memory of Oliver Heaviside. Note that $H(0)$ is not defined; it is connventional to give it the value $\frac{1}{2}$. Nonetheless, $H(x)$ is not differentiable at $x = 0$.

The shear in the preceding example is given by

$$V(x) = \frac{P}{L}\left[b - LH(x - a)\right],$$

that is, Pb/L for $x < a$ and $-Pa/L$ for $x > a$.

The step function can be used to represent loading that is distributed over a part of a beam. For example, a uniform load of intensity w distributed over the middle third of the beam is given by

$$q(x) = -w[H(x - L/3) - H(x - 2L/3)].$$

The deflection can be obtained by integrating four times, resulting in the single expression

$$v(x) = -\frac{w}{24EI}[{<}x - L/3{>}^4 - {<}x - 2L/3{>}^4] + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4,$$

and the constants $C_1, \ldots, C_4$ can be determined from the boundary conditions at $x = 0$ and $x = L$, which takes the final expression

$$EIv(x) = -\frac{w}{24}[{<}x - L/3{>}^4 - {<}x - 2L/3{>}^4] + \frac{wLx^3}{36} - \frac{wL^3x}{216}.$$

Dirac's Delta function

In order to represent the loading in the case of a concentrated force, we must be able to differentiate the step function at the point $x = 0$, which, we claimed early, is not differentiable. In order to accomplish this extrordinary task, we have to enlarge the definition of function.

Consider the shear is given by

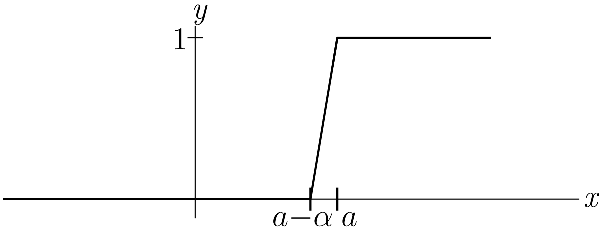
$$V(x) = \frac{P}{L}\left[b - LH(x-a)\right],$$

then the loading is

$$q(x) = \frac{d}{dx}V(x) = -P\frac{d}{dx}H(x).$$

Clearly the derivative of $H(x)$ is zero for $x \neq 0$, but at $x = 0$ it is infinite, so it is not really a function (mathematicians call it a *distribution*).

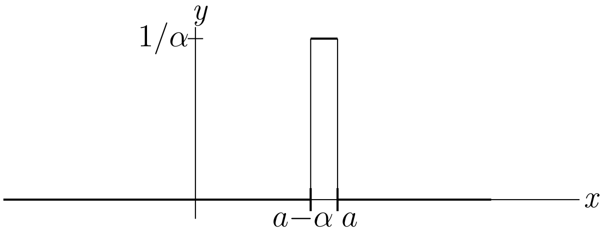
Let's consider the following approximation to the step function, say $H_\alpha(x)$:



This is given by

$$H_\alpha(x-a) = \frac{1}{\alpha}[<x-a+\alpha> - <x-a>].$$

Note that, in the limit as $\alpha \to 0$, this is just the derivative of $<x-a>$. But let's differentiate $H_\alpha(x-a)$ before taking the limit:



$$\frac{d}{dx}H_\alpha(x-a) = \frac{1}{\alpha}[H(x-a+\alpha) - H(x-a)].$$

Formally, this becomes the derivative of $H(x-a)$ in the limit as $\alpha \to 0$. What happens is that the rectangle in the picture gets narrower and taller, but its area remains 1. This limit is known as the **Dirac delta function** and is almost universally denoted $\delta(x-a)$, except that in Mechanics of Materials textbooks the strange notation $<x-a>_*^{-1}$ is used; the subscript asterisk is there to tell us that the superscipt -1 is not really the power -1. The term of Dirac's delta function is named after Paul Dirac, who was a famed British Physicist and 1933 Nobel laureate.

The loading on the beam with a concentrated load at $x=a$ can now be written as

$$q(x) = -P\delta(x-a).$$

Note that it doesn't matter what the delta function "really" means; all that matters is that when we integrate it we get the step function, and so on.

By the same token we can define the derivative of the delta function, $\delta'(x-a)$ (denoted $<x-a>_*^{-2}$ in some Mechanics of Materials textbooks). Suppose we have a downward concentrated force of magnitude M/α at $x=a$, and an upward one of the same magnitude at $x=a-\alpha$, forming the clockwise couple M, independent of α; in the limit as $\alpha \to 0$, this becomes the **concentrated** couple M. Now, mathematically, the loading is given by

$$q(x) = \frac{M}{\alpha}[\delta(x-a+\alpha) - \delta(x-a)],$$

and in the limit as $\alpha \to 0$ this **formally** becomes M times the derivative of $\delta(x-a)$, which we can write as $M\delta'(x-a)$. In some literatures, $\delta^{'}(x-a)$ is called the Doublet function. Both the doublet function and the Dirac's delta function are Singularity functions, because when $x \to 0$, both $\delta(x-a)$ and $\delta^{'}(x-a)$ become infinite, and they are only meaningful in the sense of distribution.

In summary, we list the function family associated with the singularity functions:

$$\text{Doublet function}: \quad \delta^{'}(x-a) =< x-a >_*^{-2} = \frac{d}{dx}\delta(x-a) \tag{2}$$

$$\text{Dirac's Delta function}: \quad \delta(x-a) =< x-a >_*^{-1} = \frac{d}{dx}H(x-a) \tag{3}$$

$$\text{Heaviside function}: \quad H(x-a) =< x-a >^0 = \begin{cases} 0, & x < a; \\ 1. & x > a \ . \end{cases} \tag{4}$$

$$\text{Ramp function} \quad < x-a >^1 = \begin{cases} 0, & x < a; \\ (x-a) & x \geq a \ . \end{cases} \tag{5}$$

$$\text{Bracket power function} \quad < x-a >^n = \begin{cases} 0, & x < a; \\ (x-a)^n & x \geq a \ . \end{cases} \quad n \geq 1 \tag{6}$$