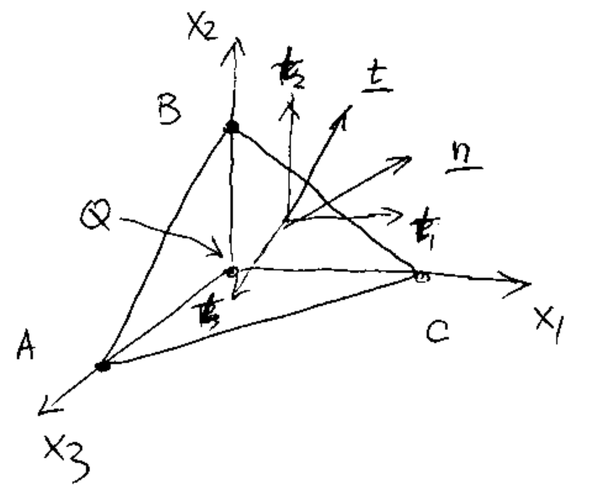
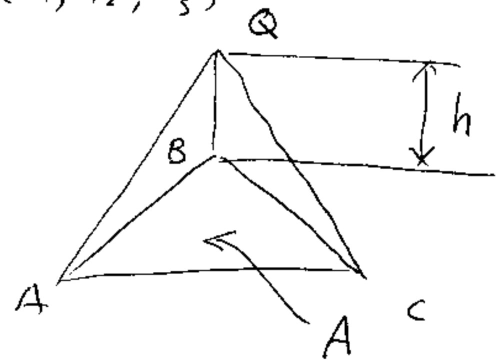
Lecture 2 Cauchy's Formula

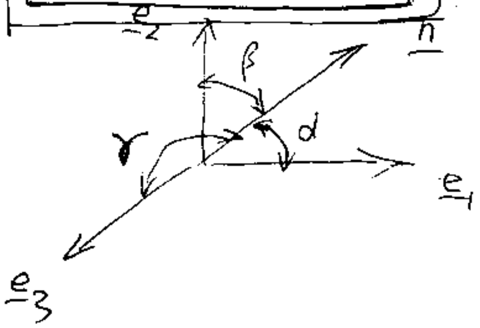
Consider an infinitesimal tetrahedron that is cut from a continuum at the point $Q\ (X, Y, Z)$ or $Q\ (x_1, x_2, x_3)$





$$\Delta V = \frac{1}{3} hA$$

The unit vector $\underline{n}$



2 ways

$$\cos\alpha = \cos(\underline{n}, \underline{e}_1) = l = n_1$$

$$\cos\beta = \cos(\underline{n}, \underline{e}_2) = m = n_2$$

$$\cos\gamma = \cos(\underline{n}, \underline{e}_3) = n = n_3$$

And $\underline{n} = l\,\underline{e}_1 + m\,\underline{e}_2 + n\,\underline{e}_3 = n_i\,\underline{e}_i$

note that $|\underline{n}|^2 = l^2 + m^2 + n^2 = 1$

Consider an oriented plane segment

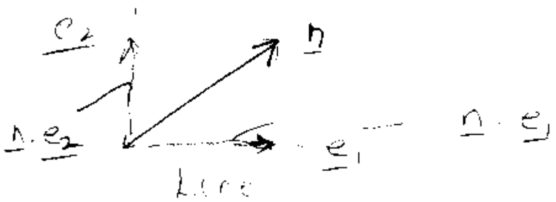
$$\underline{A} = A\,\underline{n} = A\,(l\,\underline{e}_1 + m\,\underline{e}_2 + n\,\underline{e}_2)$$

Analogy

$$A_1 = A_{QAB} = \underline{A} \cdot \underline{e}_1 = A\,(l\,\underline{e}_1 + m\,\underline{e}_2 + n\,\underline{e}_3) \cdot \underline{e}_1 = Al = An_1$$

$$A_2 = A_{QAC} = \underline{A} \cdot \underline{e}_2 = A\,(l\,\underline{e}_1 + m\,\underline{e}_2 + n\,\underline{e}_3) \cdot \underline{e}_2 = Am = An_2$$

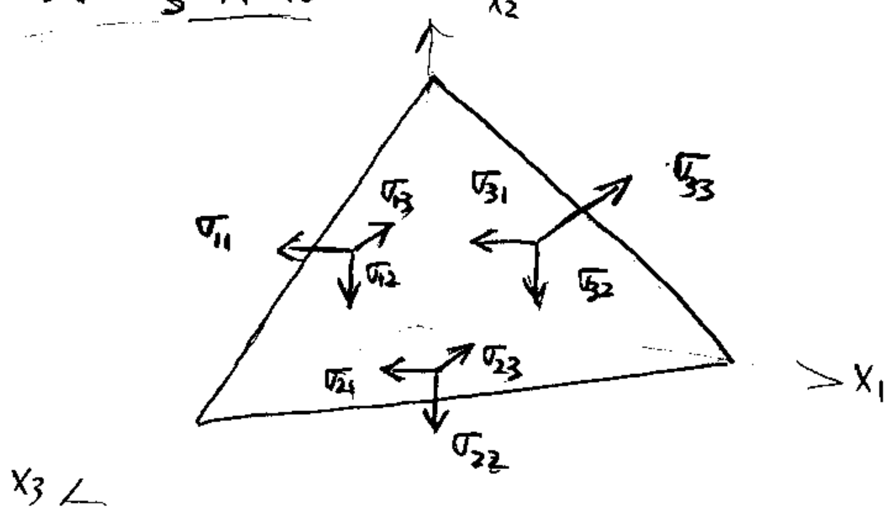
$$A_3 = A_{QBC} = \underline{A} \cdot \underline{e}_3 = A\,(l\,\underline{e}_1 + m\,\underline{e}_2 + n\,\underline{e}_3) \cdot \underline{e}_3 = An = An_3$$



$$\Sigma F_x = \Sigma F_1 = 0$$

$$-\sigma_{11} \, A\ell - \sigma_{21} \, Am - \sigma_{31} \, An + t_1 A + f_1 \Delta V = 0$$

$$\Delta V = \frac{1}{3} A \, h$$



$\Sigma F_2 = 0,$

$$-\sigma_{12} \, A\ell - \sigma_{22} \, Am - \sigma_{32} \, An + t_2 A + f_2 \overset{\Delta V}{\boxed{\frac{1}{3} A h}} = 0$$

$\Sigma F_3 = 0$

$$-\sigma_{13} \, A\ell - \sigma_{23} \, Am - \sigma_{33} \, An + t_3 A + f_3 \left(\frac{1}{3} A h\right) = 0$$

$$-\sigma_{11} A n_1 - \sigma_{21} A n_2 - \sigma_{31} A n_3 + t_1 A + f_1 \left(\tfrac{1}{3} A h\right) = 0$$

$$-\sigma_{12} A n_1 - \sigma_{22} A n_2 - \sigma_{32} A n_3 + t_2 A + f_2 \left(\tfrac{1}{3} A h\right) = 0$$

$$-\sigma_{13} A n_1 - \sigma_{23} A n_3 - \sigma_{33} A n_3 + t_3 A + f_3 \left(\tfrac{1}{3} A h\right) = 0$$

when $\Delta V \longrightarrow 0$, $A \longrightarrow 0$ and $h \longrightarrow 0$

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 = \sigma_{j1} n_j$$

$$t_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3 = \sigma_{j2} n_j$$

$$t_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 = \sigma_{j3} n_j$$

In summary:

$$\boxed{t_i = \sigma_{ji} n_j}$$

or

$$\boxed{P_i = \sigma_{ji} n_j} \longleftarrow \boxed{\text{Cauchy's formula}}$$

the normal of the inclined plane

Note that

$$(n_1, n_2, n_3) = (l, m, n)$$

Many people write

$$\boxed{t_i = \sigma_{ij} n_j} \quad X \longleftarrow$$

it is O.K, because $\sigma_{ij} = \sigma_{ji}$, this is true only for Cauchy stress

it should be

$$\boxed{t_i = \sigma_{ji} n_j} \quad V$$

GENERAL

Pro

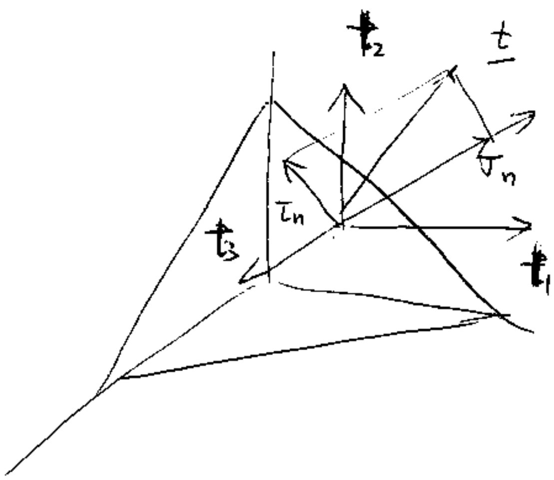
Matrix notation

$$[t_i] = [\sigma_{ij}^T][n_j]$$

← Note that in matrix notation the summation index or indicies has to be $\boxed{\text{adjacent}}$ i.e.

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11}^T & \sigma_{12}^T & \sigma_{13}^T \\ \sigma_{21}^T & \sigma_{22} & \sigma_{23}^T \\ \sigma_{31}^T & \sigma_{32}^T & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \quad \text{or :} \quad \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

Applications of Cauchy's formula : Find the normal stress and shear stress on an inclined plane. σ_n ? & τ_n ?



the normal stress should be.

$$\sigma_n = \underline{t} \cdot \underline{n}$$

$$\sigma_n = t_1 n_1 + t_2 n_2 + t_3 n_3$$

$$= t_i n_i = (\sigma_{ji} n_j) n_i$$

shear stress

$$\tau_n = \sqrt{|\underline{t}|^2 - \sigma_n^2}$$

$$= \sqrt{(t_1^2 + t_2^2 + t_3^2) - (t_1 n_1 + t_2 n_2 + t_3 n_3)^2}$$