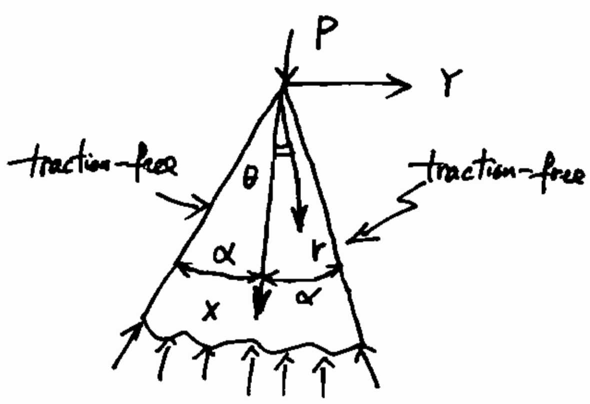
Lecture 10 Stresses due to concentrated load

Problem 10-1

study stress state at the point where concentrated load is applied

Consider a concentrated force P acting at the vertex of a semi-infinite wedge.

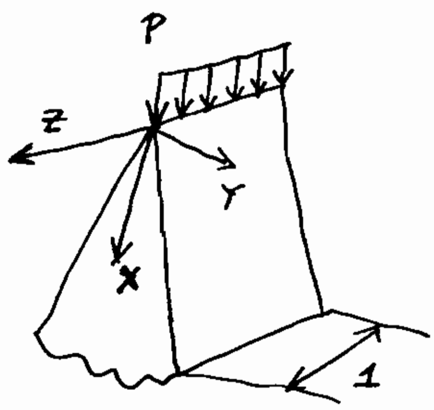


The problem is symmetric with axis x.

Note that P is not force, it is a load distribution.

It has the unit : $P \longleftrightarrow \dfrac{[\text{force}]}{[\text{length}]}$

The thickness of the wedge is taken as unity, so P is the load per unit thickness.



Use the semi-inverse method

$$P \longrightarrow \frac{[\text{force}]}{[\text{length}]} = \frac{F}{L}$$

$$\sigma \longrightarrow \frac{[\text{force}]}{[\text{length}]^2} = \frac{F}{L^2} \rightarrow \left(\frac{P}{r}\right)$$

$$\phi \longrightarrow \sigma \cdot [\text{length}]^2 = Pr \boxed{f(\theta)}$$

dimensionless

$$\left(\text{because } \sigma = \frac{\phi}{[\text{length}]^2}\right)$$

A reasonable assumption is that

$$\phi = Pr f(\theta)$$

Our task is to determine $f(\theta)$

Now consider bi-harmonic equation,

$$\boxed{\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \phi = 0}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) Pr f(\theta)$$

$$= \frac{P}{r} f(\theta) + \frac{P}{r}\frac{d^2 f}{d\theta^2} = P\left(\frac{1}{r} f(\theta) + \frac{1}{r}\frac{d^2 f}{d\theta^2}\right)$$

$$= \frac{P}{r}\left(f(\theta) + \frac{d^2 f}{d\theta^2}\right)$$

$$\nabla^2\nabla^2\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{P}{r}\right)\left(f(\theta) + \frac{d^2 f}{d\theta^2}\right)$$

$$= \frac{2P}{r^3}\left(f(\theta) + \frac{d^2 f}{d\theta^2}\right) - \frac{P}{r^3}\left(f(\theta) + \frac{d^2 f}{d\theta^2}\right) + \frac{P}{r^3}\left(\frac{d^2 f}{d\theta^2} + \frac{d^4 f}{d\theta^4}\right)$$

$$= \frac{P}{r^3}\left(f(\theta) + \frac{d^2 f}{d\theta^2}\right) + \frac{P}{r^3}\left(\frac{d^2 f}{d\theta^2} + \frac{d^4 f}{d\theta^4}\right)$$

$$= \frac{P}{r^3}\left(\frac{d^4 f}{d\theta^4} + 2\frac{d^2 f}{d\theta^2} + f(\theta)\right) = 0$$

Let $f = A\exp(a\theta) \Rightarrow$ characteristic equation

$$a^4 + 2a^2 + 1 = 0 \Rightarrow (a^2+1)^2 = 0$$

$$\Rightarrow a^2 = -1, \Rightarrow a = \pm i$$

$\boxed{\text{double roots}}$

Repeated roots $a_1 = i, \quad a_2 = i, \quad a_3 = -i, \quad a_4 = -i$

Therefore

$$f(\theta) = A\cos\theta + B\sin\theta + \theta(D\cos\theta + E\sin\theta)$$

$$\phi = Prf(\theta) = P(Ar\cos\theta + Br\sin\theta + r\theta(D\cos\theta + E\sin\theta))$$

$$= P(Ax + By + r\theta(D\cos\theta + E\sin\theta))$$

$PAx + PBx \longrightarrow$ will not cause any stresses

$$\phi = Pr\theta\,(C\sin\theta + D\cos\theta)$$

But ϕ is symmetric with x-axis, i.e. ϕ should be even in θ

$$\boxed{\phi = Pc\, r\theta\sin\theta} \qquad \boxed{\text{Eq. (a) page 118}}$$

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

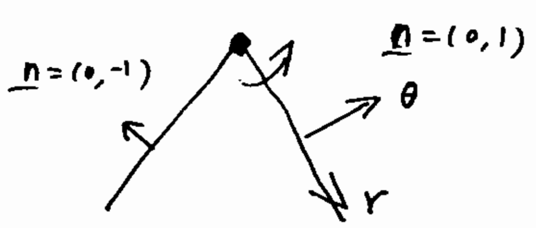
$$= Pc\left[\frac{\theta\sin\theta}{r} + \frac{1}{r}\frac{\partial^2}{\partial\theta^2}(\theta\sin\theta)\right]$$

$$= Pc\left[\frac{\theta\sin\theta}{r} + \frac{2}{r}\cos\theta - \frac{\theta\sin\theta}{r}\right] = \frac{2Pc}{r}\cos\theta$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} = 0$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 0 \qquad \underline{n} = \begin{pmatrix} n_r \\ n_\theta \end{pmatrix}$$

How to determine C ? B.C. ?



$\underline{n} = (l, m)$

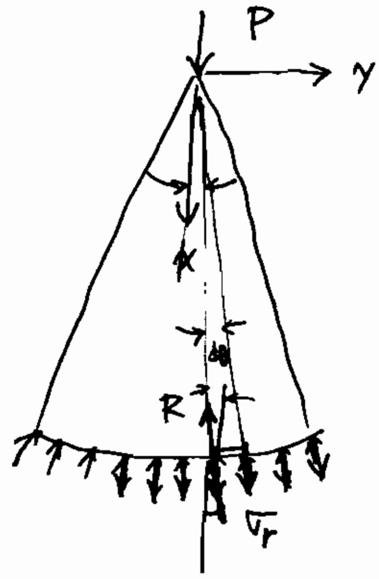
$l = \cos(\underline{n}, r) = 0$

$m = \cos(\underline{n}, \theta) = \pm 1$

By
$$\begin{cases} l\sigma_r + m\sigma_{r\theta} = \bar{t}_r = 0 \\ l\sigma_{r\theta} + m\sigma_\theta = \bar{t}_\theta = 0 \end{cases} \Rightarrow \begin{cases} \sigma_{r\theta}\big|_{\theta=\pm\alpha} = 0 \\ \sigma_\theta\big|_{\theta=\pm\alpha} = 0 \end{cases}$$

Automatically satisfied !

$$\begin{bmatrix} t_r \\ t_\theta \end{bmatrix} = \begin{bmatrix} \sigma_r & \sigma_{r\theta} \\ \sigma_{\theta r} & \sigma_\theta \end{bmatrix} \begin{bmatrix} 0 \\ \pm 1 \end{bmatrix}$$



Forage

$\Sigma F_x = 0$

Remote B.C

$$\int_{2\alpha} \sigma_r \Big|_{r=R} \cos\theta \,(R\,d\theta) + P = 0$$

$$\sigma_r = \frac{2PC}{r}\cos\theta$$

$$\Rightarrow \int_{-\alpha}^{\alpha} \sigma_r \Big|_{r=R} \cos\theta \, R\, d\theta + P = 0$$

$$\frac{2\cancel{P}C}{\cancel{R}} \int_{-\alpha}^{\alpha} \cancel{R} \cos^2\theta \, d\theta + \cancel{P} = 0$$

$$2C \int_{-\alpha}^{\alpha} \frac{1}{2}(1+\cos 2\theta)\, d\theta + 1 = 0$$

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$C\left[\theta + \frac{1}{2}\sin 2\theta\right]\Big|_{-\alpha}^{\alpha} + 1 = 0$$

$$C\left[2\alpha + \sin 2\alpha\right] + 1 = 0$$

$$(\theta\sin\theta)'' = (\sin\theta + \theta\cos\theta)' = \cos\theta + \theta\cos\theta - \theta\sin\theta$$

$$\boxed{C = -\frac{1}{2}\,\frac{1}{\left[\alpha + \frac{1}{2}\sin 2\alpha\right]}}$$

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$\rightarrow 2\cos\theta$
$-\theta\sin\theta$

$$\boxed{\phi = -\frac{P\, r\, \theta \sin\theta}{2\left[\alpha + \frac{1}{2}\sin 2\alpha\right]}} \leftarrow$$

This solution is due to J. H. Michell

↑ An australian

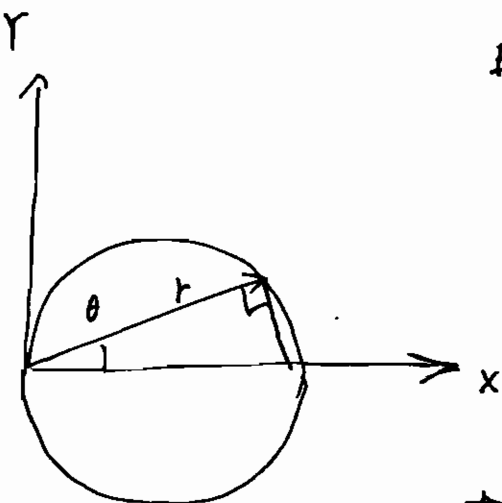
$$\boxed{\sigma_r = -\frac{P\,\cancel{\theta}\cos\theta}{r\left[\alpha + \frac{1}{2}\sin 2\alpha\right]}}$$

$$\boxed{\sigma_\theta = 0, \quad \sigma_{r\theta} = 0}$$

Speaial case : $\alpha = \frac{\pi}{2}$ $\quad \sin 2\alpha = \sin \pi = 0$

$$\boxed{\begin{aligned} \phi &= -\frac{P r \theta \sin\theta}{\pi} \\ \sigma_r &= -\frac{2P}{\pi}\frac{\cos\theta}{r} \end{aligned}}$$

$\sigma_\theta = 0, \quad \sigma_{r\theta} = 0$



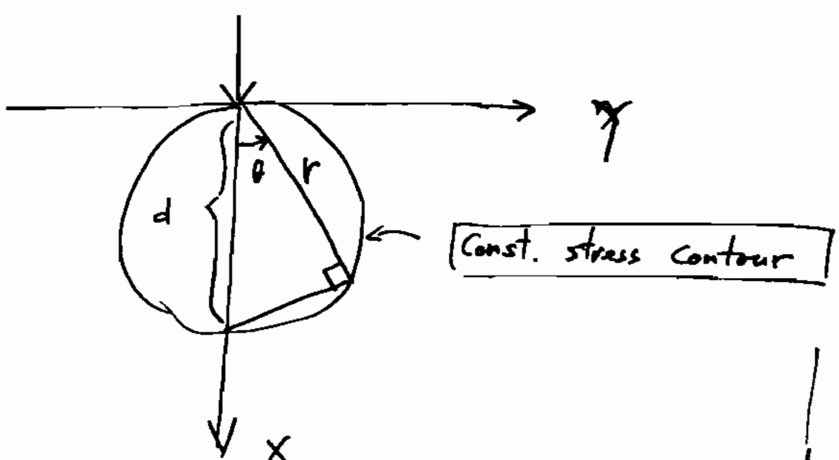
Equation of a circle in polar coordinate

$$\boxed{d \cos\theta = r}$$

$\Rightarrow \quad d = \frac{r}{\cos\theta} \Rightarrow \left(\frac{\cos\theta}{r}\right) = \frac{1}{d}$

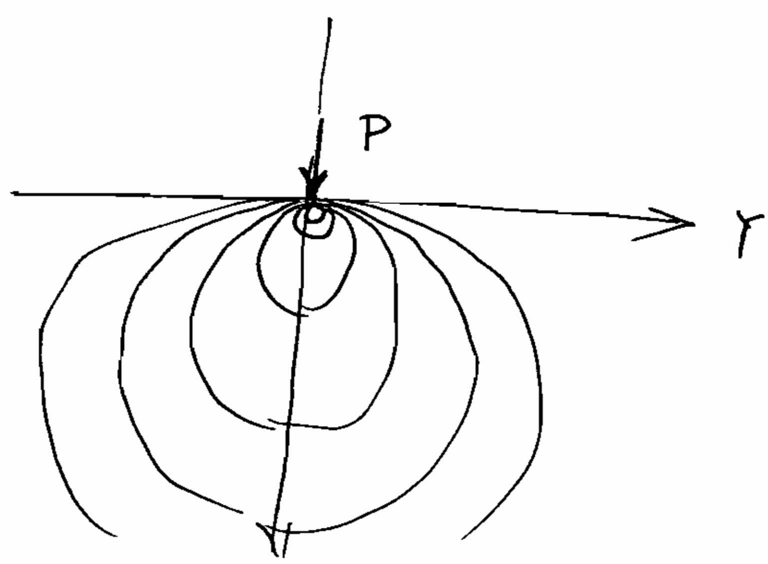
$\boxed{\sigma_r = -\frac{2P}{\pi}\frac{1}{d}}$

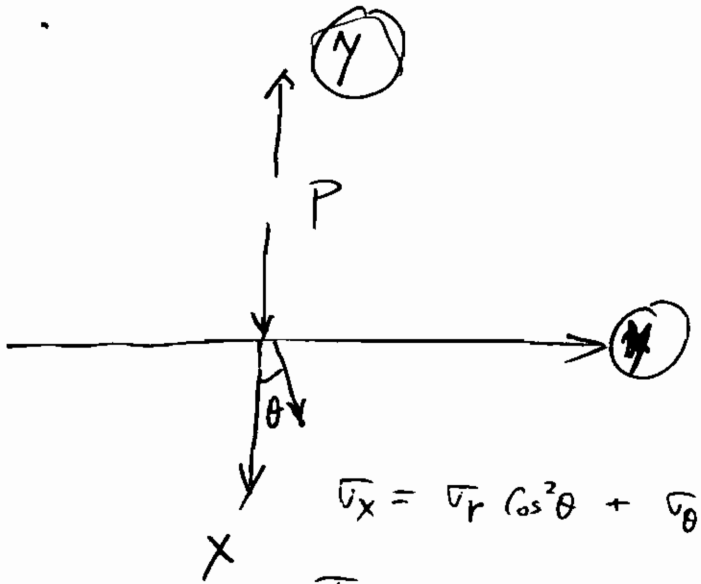
$\Rightarrow \quad \boxed{\sigma_r = -\frac{2P}{\pi d} = \text{Const.}}$



$d \to 0$

$\sigma_r \to \infty$





$$\sigma_x = \sigma_r \cos^2\theta + \sigma_\theta \sin^2\theta - 2\tau_{r\theta} \sin\theta\cos\theta$$

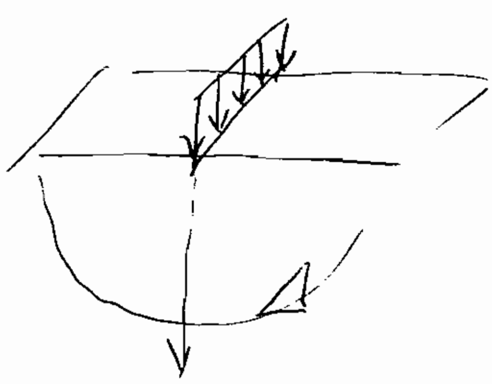
$$\tau_{xy} = (\sigma_r - \sigma_\theta)\sin\theta\cos\theta + \tau_{r\theta}(\cos^2\theta - \sin^2\theta)$$

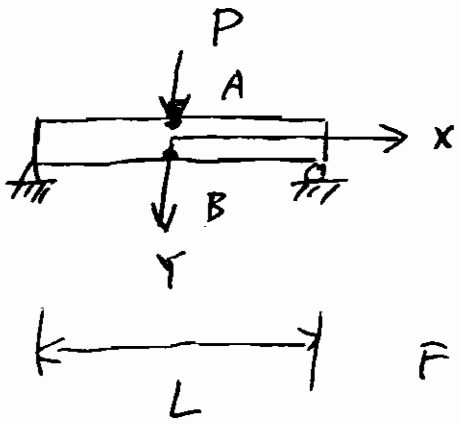
$$\sigma_y = \sigma_r \sin^2\theta + \sigma_\theta \cos^2\theta + 2\tau_{r\theta}\sin\theta\cos\theta$$

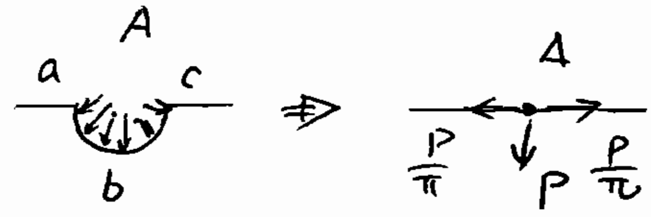
Since $\sigma_\theta = \tau_{r\theta} = 0$

$$\left\{\begin{aligned} \sigma_x &= \sigma_r \cos^2\theta = -\frac{2P}{\pi}\frac{\cos\theta}{r}\cdot\cos^2\theta \\ &= -\frac{2P}{\pi}\frac{\cos^3\theta}{r} \\ \sigma_y &= \sigma_r \sin^2\theta = -\frac{2P}{\pi}\frac{\cos\theta\sin^2\theta}{r} \end{aligned}\right.$$

$$\tau_{xy} = \sigma_r \sin\theta\cos\theta = -\frac{2P}{\pi r}\frac{\cos^2\theta\sin\theta}{r}$$





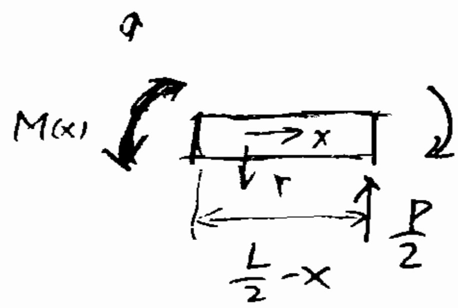


$$F = \int_0^{\frac{\pi}{2}} (\sigma_r \sin\theta)\, R\, d\theta$$

$$= \frac{2P}{\pi} \int_0^{\pi/2} \cancel{R} \frac{\cos\theta}{\cancel{R}} \sin\theta \, d\theta$$

$$= \frac{2P}{\pi} \; \frac{1}{2} \sin^2\theta \Big|_0^{\pi/2} = \frac{P}{\pi}$$

Beam theory



$$\sigma_x' = -\frac{My}{I}$$

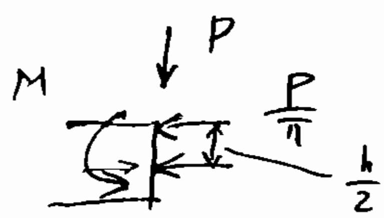
$$M_z(x) = -\frac{P}{2}\left(\frac{L}{2} - x\right)$$

$$\Sigma M_x = 0$$

$$+M(x) + \frac{P}{2}\left(\frac{L}{2} - x\right) = 0$$

$$\sigma_x' = +\frac{P}{2I}\left(\frac{L}{2} - x\right) y$$

$$I = \frac{bh^3}{12}$$



$$\sigma_x' = +\frac{6P}{bh^3}\left(\frac{L}{2} - x\right) y$$

$$M = \frac{Ph}{2\pi}$$

$$\sigma_x^c = +\frac{6P}{bh^3}\left(\frac{L}{2} - x\right) y - \underbrace{\frac{Ph\,y}{2\pi I}}_{\frac{Py}{6\pi b h^2}} - \frac{P}{\pi b h}$$

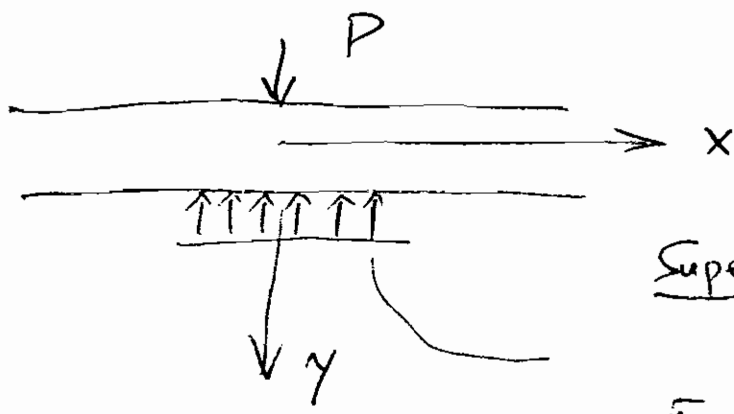
At. $y = -\frac{h}{2}$, $x = 0$

$$(\sigma_x)_A = \frac{6P}{bh^3}\left(\frac{L}{2}\right) \overset{(-\frac{h}{2})}{y} - \frac{Ph\left(-\frac{h}{2}\right)}{6\pi bh^3} - \frac{P}{\pi bh}$$

$$(\sigma_x)_A = \frac{P}{12bh\pi} - \frac{3PL}{2bh^3} - \frac{P}{\pi bh}$$

$$(\sigma_x)_B = \frac{3PL}{2bh^3} - \frac{P}{12\pi bh} - \frac{P}{\pi bh} = \boxed{\frac{3PL}{2bh^3} - 0.637\frac{P}{bh}}$$

$(\sigma_x)_B$ is more complex

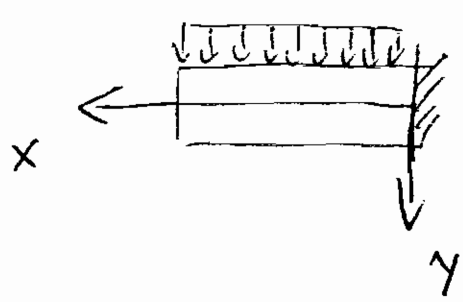


Superposition

$$\sigma_{yy} = \sigma_{xx} \sim \sigma_r \quad (\theta = 0)$$

$$\sim -\frac{2P}{\pi}\frac{\cos^3\theta}{r}$$

$$\sim -\frac{2P}{\pi r} = \frac{-2P}{\pi h}$$



$$\sigma_{xx} = -\frac{P}{2I}x^2y - \frac{Ph^2y}{5I} + \frac{P}{3I}y^3 \qquad \frac{h}{2}$$

$$\boxed{p = \frac{2P}{\pi b}}$$

$$\sigma_{yy} = -\frac{P}{6I}(y^3 - 3h^2y + 2h^3)$$

$$\sigma_{xy} = \frac{Px}{2I}(y^2 - h^2)$$

$h \rightarrow 2h$

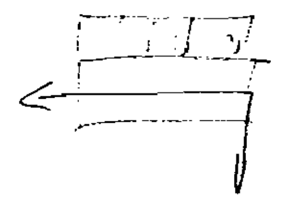
$$(\sigma_{xx})_B = (\sigma_x)_B' - \left(\frac{2P}{\pi b}\right)\frac{y^3}{3I} + \frac{P(\frac{h}{2})^2 y}{5I}$$

$\frac{bh^3}{12}$ $\frac{bh^5}{12}$

$$= (\sigma_x)_B' - 2Py^3 + \frac{3Py}{5bh}$$

$$\left(\frac{2P}{\pi b}\right)\frac{(\frac{h}{2})^3}{\frac{bh^3}{4}}$$

$$= \left(\frac{8P}{\pi b}\right)\frac{\cancel{h^3}}{8b\cancel{h^3}} = \frac{P}{\pi b^2}$$



charge

$$\sigma_{xx} = -\frac{Ph^2 y}{5I} + \frac{P}{3I}y^3 \qquad I = \frac{2h^3}{3}$$

$$= -\frac{Ph^2 y}{5\times\frac{2h^3}{3}} + \frac{P}{\cancel{3}\times\frac{2h^3}{\cancel{3}}}y^3$$

$$\boxed{\beta = \frac{2P}{\pi h}}$$

$$\boxed{\sigma_{xx} = -\frac{3\beta y}{10h} + \frac{\beta y^3}{2h^3}}$$

$$\boxed{h = \frac{h}{2}}$$

P.3.19

$$\sigma_{xx} = -\frac{P5x^2 y}{10I} - \frac{Ph^2 y}{5I} + \frac{Py^3}{3I}$$

$$\boxed{I = \frac{2h^3}{3}}$$

$$(\sigma_x)_B = (\sigma_x)_B' + \left(\frac{2P}{\pi h}\right)\left[+ \frac{(h/2)^3}{2(h/2)^3} - \frac{3(\frac{h}{2})}{10(\frac{h}{2})}\right] \qquad b=1$$

$$= (\sigma_x)_B' + \left(\frac{2P}{\pi h b}\right)\left[\frac{1}{2} - \frac{3}{10}\right]$$

$$(\sigma_x)_B = \frac{3PL}{2bh^2} - 0.508\frac{P}{bh}$$