Lecture 17 Pre-stress design

Lamé's solution (Eq. (8.8)), (by French engineer G. Lamé)
Recall Lamé constants, λ & μ

$$\begin{cases} \sigma_r = \dfrac{a^2 P_i - b^2 P_o}{b^2-a^2} - \dfrac{(P_i - P_o)\, a^2 b^2}{(b^2-a^2) r^2} & (*1) \\[2ex] \sigma_\theta = \dfrac{a^2 P_i - b^2 P_o}{b^2-a^2} + \dfrac{(P_i - P_o)\, a^2 b^2}{(b^2-a^2) r^2} & (*2) \\[2ex] u = \dfrac{(1-\nu)}{E} \dfrac{(a^2 P_i - b^2 P_o) r}{b^2-a^2} + \dfrac{(1+\nu)}{E} \dfrac{(P_i - P_o)\, a^2 b^2}{(b^2-a^2)\, r} & (*3) \end{cases}$$

Suppose the main purpose of a thick cylinder is to hold internal pressure.
Then, $P_o = 0$ at $r = a$, both σ_r & σ_θ are at their maximum

$$\begin{cases} \sigma_r \big|_{r=a} = -P_i \\[2ex] \sigma_\theta \big|_{r=a} = \left(\dfrac{R^2+1}{R^2-1} \right) P_i \end{cases},$$

$$\sigma_\theta = \frac{a^2 P_i}{b^2-a^2}\left(1 + \frac{b^2}{r^2}\right) \quad (*2b)$$

Inner pressure only

$$\sigma_r = \frac{a^2 P_i}{b^2-a^2}\left(1 - \frac{b^2}{r^2}\right) \quad (*1b)$$

where $R = \dfrac{b}{a}$

By increasing the thickness of the cylinder $b \to \infty$

$$\sigma_\theta \big|_{r=a} = P_i \quad \leftarrow \text{It does not do much!}$$

$\boxed{\text{How can we reduce } \sigma_\theta \big|_{r=a}}$?

Re-examine (#2), at $r = a$

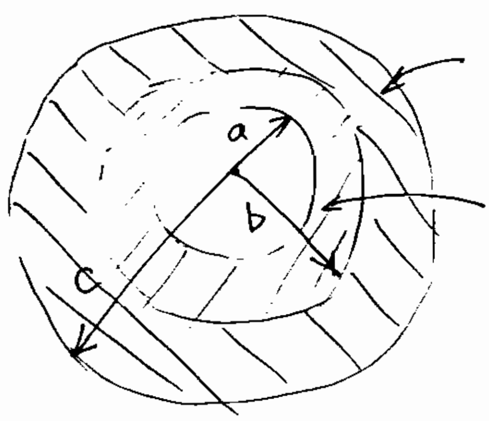
$$\sigma_\theta = (P_i - P_o)\left(\frac{1+R^2}{R^2-1}\right) - P_o$$

Obviously, if there is an external pressure, σ_θ will be or can be reduced significantly. This is to say that: (suggests)

If somehow, we can <u>pre-stress</u> or w can apply external pressure pre-stress the cylinder, the cylinder may be able to take <u>higher internal pressure</u>. $\longleftrightarrow$ $\boxed{\text{The question is HOW ?}}$ $\Longrightarrow$ $\boxed{\text{pre-stress design}}$

An efficient pre-stress design for thick cylinder is to make compound cylinders with radial misfit to create a pre-stress state that can later on compensate the internal pressure loading.

Consider a compound cylinder made of two cylinders

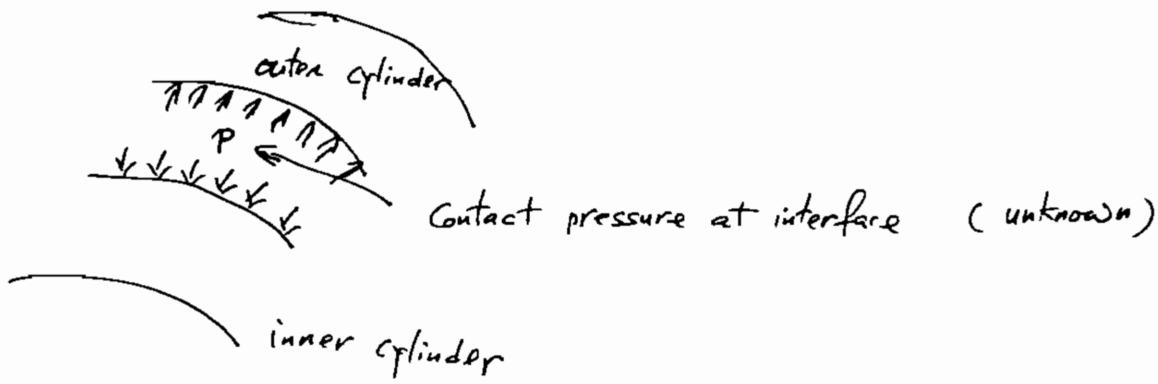


Define $R_1 = \frac{b}{a}$, $R_2 = \frac{c}{a}$

To create pre-stress state, there is a radial misfit between two cylinders: the inner radius of the outer cylinder, r_i^{OC}, is smaller than the outer radius of the inner cylinder r_o^{IC}, i.e.

$$\boxed{\delta = r_o^{IC} - r_i^{OC}} \longleftarrow \text{Shrinking allowance}$$

This will create a statically indeterminate system, and the inner cylinder will be under external pressure, while the outer cylinder will be under internal pressure. It will therefore provide stress relief at $r=a$ i.e. the inner surface of the inner cylinder.



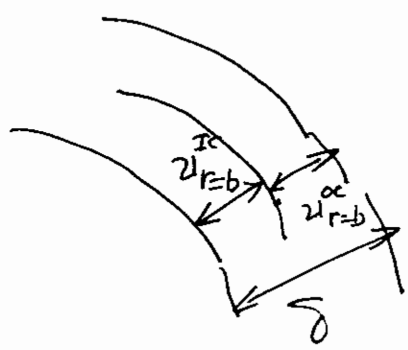
To find the exact pre-stress state, we have to solve the statically indeterminate problem

In the prestress design, the shrinking allowance is based on how much external pressure you want to apply to the inner cylinder.

To relate $\delta - p$, we use the compatibility condition at the interface,

$$\delta = |u^{IC}_{r=b}| + |u^{OC}|_{r=b}|$$

$(P_o = P, \quad P_i = 0)$



Under external pressure p, the inner cylinder has displacement at $r=b$

$$u^{IC}\Big|_{r=b} = \frac{(1-\nu)}{E}\frac{(-b^2 p)b}{b^2-a^2} + \frac{(1+\nu)}{E}\frac{(-p)a^2b^2}{(b^2-a^2)b}$$

$$= -\frac{(1-\nu)}{E}\frac{b^3 p}{b^2-a^2} - \frac{(1+\nu)}{E}\frac{a^2 b p}{b^2-a^2}$$

$$= -\frac{b^3 p}{E(b^2-a^2)}\left((1-\nu) + (1+\nu)\frac{a^2}{b^2}\right)$$

$$\left| u^{IC}\Big|_{r=b} \right| = \left| -\frac{b^3 p}{E(b^2-a^2)} \left((1-\nu) + (1+\nu)\frac{a^2}{b^2} \right) \right|$$

$$= \frac{bp}{E}\left(\frac{a^2+b^2}{b^2-a^2} - \nu \right)$$

Meanwhile, the outer cylinder is under internal pressure.

($a \rightarrow b$, $b \rightarrow c$) In Eq (*3)

The radial displacement at $r=b$, will be ($p_i = p$, $p_o = 0$)

$$u^{OC}\Big|_{r=b} = \frac{(1-\nu)}{E}\frac{b^3 p}{c^2-b^2} + \frac{(1+\nu)}{E}\frac{p\, b^2 c^2}{(c^2-b^2)b}$$

$$= \frac{bp}{E}\left(\frac{c^2+b^2}{c^2-b^2} + \nu \right)$$

Based on compatibility condition.

$$\delta = \left| u^{IC}_{r=b} \right| + \left| u^{OC}_{r=b} \right|$$

$$= \frac{bp}{E}\left(\frac{a^2+b^2}{b^2-a^2} \cancelto{0}{-\nu} \right) + \frac{bp}{E}\left(\frac{c^2+b^2}{c^2-b^2} \cancelto{0}{+\nu} \right)$$

$\Longrightarrow$ We now can relate $p \sim \delta$

$$\boxed{p = \left(\frac{E\delta}{b} \right) \frac{(b^2-a^2)(c^2-b^2)}{2b^2(c^2-a^2)}}$$

$\uparrow$ $\boxed{\text{interfacial pressure}}$

Once we determined interfacial pressure, we can determine the pre-stress state :

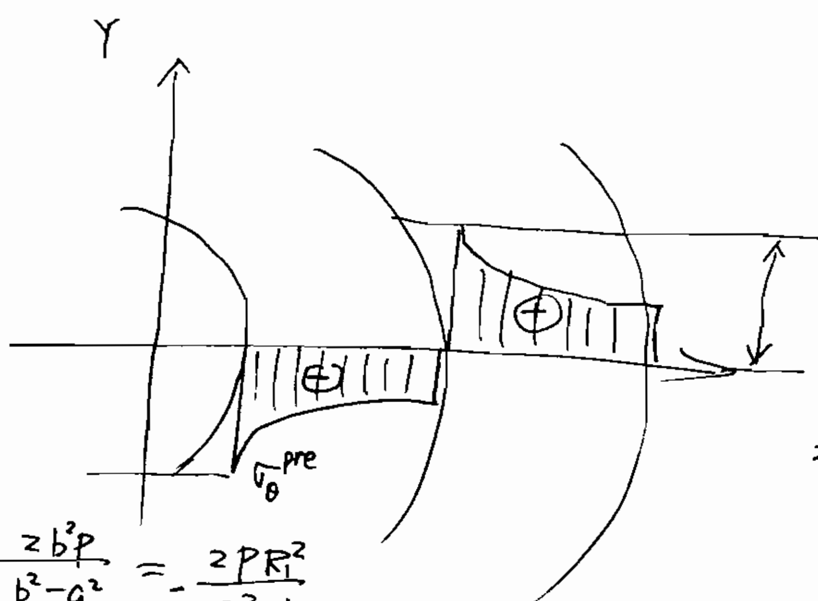
For inner cylinder $P_i = 0, \quad P_o = P$

$$\sigma_\theta^{pre(Ic)} = -\frac{b^2 P}{b^2-a^2} - \frac{P a^2 b^2}{(b^2-a^2) r^2} = -\frac{b^2 P}{(b^2-a^2)}\left(1+\frac{a^2}{r^2}\right) < 0$$

For outer cylinder $P_i = P, \quad P_o = 0 \qquad (a \to b, \quad b \to c)$

$$\sigma_\theta^{pre(oc)} = \frac{b^2 P}{c^2-b^2} + \frac{b^2 c^2 P}{(c^2-b^2) r^2}$$

$$= \frac{b^2 P}{c^2-b^2}\left(1+\frac{c^2}{r^2}\right) > 0$$



$$\frac{b^2 P}{c^2-b^2}\left(1+\frac{c^2}{b^2}\right) = P\left(\frac{R_2^2+1}{R_2^2-1}\right)$$

$$-\frac{2b^2 P}{b^2-a^2} = -\frac{2PR_1^2}{R_1^2-1}$$

So far, there is no loading yet. This is pre-stress state

Now, we load the compound cylinder and the functional internal pressure is denoted as $P_i = P_\ell$ ← loading

Then the total tangential stress should be the superposition of pre-stress and loading stress, i.e.

$$\sigma_\theta^t = \sigma_\theta^{pre} + \sigma_\theta^{load}$$

σ_θ^{load} is calculated based on Eq (*2) or (*2b) by letting

$a \to a$, $b \to c$ ← the compound cylinder

$P_i = P_\ell$

Therefore

$$\sigma_\theta^t = \underbrace{\frac{a^2 P_\ell}{c^2-a^2}\left(1+\frac{c^2}{r^2}\right)}_{\sigma_\theta^{load}} + \begin{cases} -\dfrac{b^2 p}{b^2-a^2}\left(1+\dfrac{a^2}{r^2}\right), & a \le r \le b \\ \dfrac{b^2 p}{c^2-b^2}\left(1+\dfrac{c^2}{r^2}\right), & b \le r \le c \end{cases}$$

Example 8.3

$a = 150$ mm, $b = 200$ mm, $c = 250$ mm,

$E = 200$ GPa, $\delta = 0.1$ mm. subject to internal pressure

$P_i = 140$ MPa

Determine the overall tangential stress distribution.

[Solution]

@ step 1 : prestress :

$$\sigma_\theta^{pre} = \begin{cases} -\dfrac{b^2 p}{b^2+a^2}\left(1+\dfrac{a^2}{r^2}\right), & a \le r \le b \\ \dfrac{b^2 p}{c^2-b^2}\left(1+\dfrac{c^2}{r^2}\right), & b \le r \le c \end{cases}$$

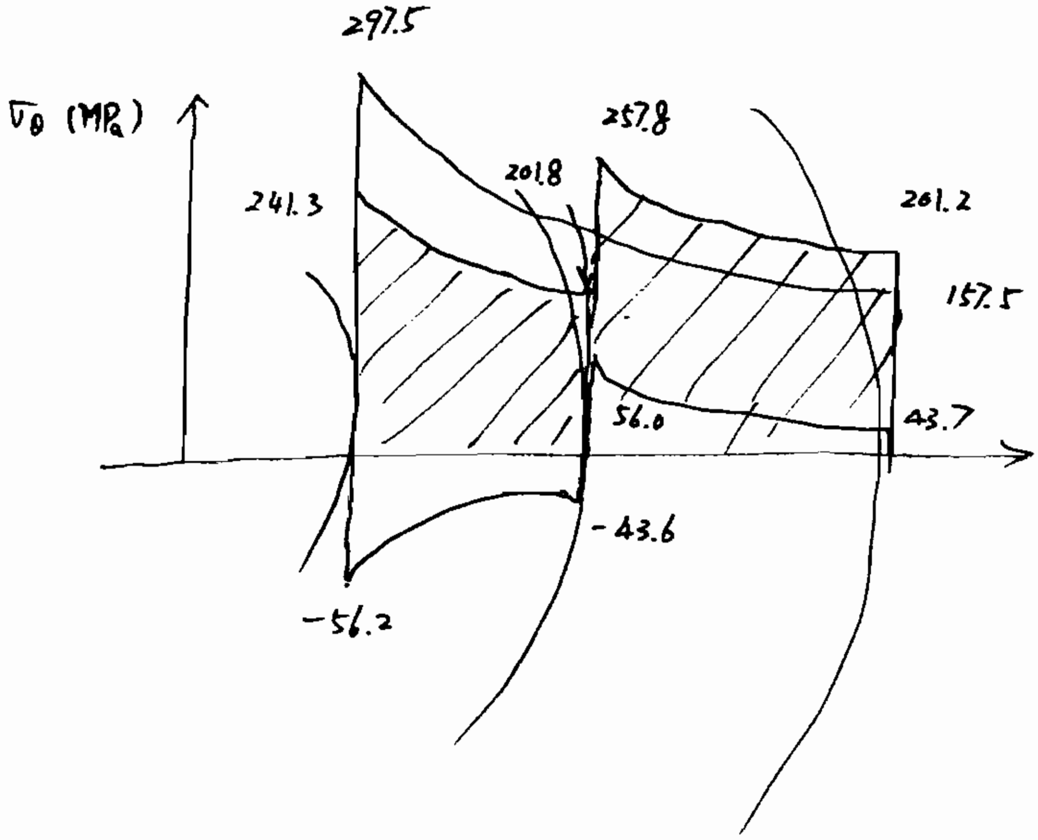
$$p = \left(\frac{E\delta}{b}\right)\frac{(b^2-a^2)(c^2-b^2)}{2b^2(c^2-a^2)}$$

$$= \frac{(200\times 10^9)\times 10^{-4}}{0.2}\;\frac{(0.2^2-0.15^2)(0.25^2-0.2^2)}{2\times(0.2)^2(0.25^2-0.15^2)} = 12.3 \text{ MPa}$$

In the inner cylinder

$$\sigma_\theta^{pre}\Big|_{r=a=0.15} = -\frac{2pb^2}{b^2-a^2} = -\frac{2(12.3\times 10^6)(0.2^2)}{0.2^2-0.15^2} = -56.2 \text{ MPa}$$

$$\sigma_\theta^{pre}\Big|_{r=b=0.2} = -p\frac{b^2+a^2}{b^2-a^2} = -(12.3\times 10^6)\frac{0.2^2+0.15^2}{0.2^2-0.15^2} = -43.9 \text{ MPa}$$



Outer cylinder:

$$\sigma_\theta^{pre}\Big|_{r=b=0.2} = p\,\frac{b^2+c^2}{c^2-b^2} = (12.3\times10^6)\,\frac{0.2^2+0.25^2}{0.25^2-0.2^2} = 56.0\ MPa$$

$$\sigma_\theta^{pre}\Big|_{r=c=0.25} = \frac{2pb^2}{c^2-b^2} = \frac{2\times(12.3\times10^6)\,(0.2)^2}{0.25^2-0.2^2} = 43.7\ MPa$$

(2) $\sigma_\theta^{\cancel{reg}\ load}$

$$\sigma_\theta^{\cancel{reg}\,l} = \frac{a^2 p_i}{c^2-a^2}\left(1+\frac{c^2}{r^2}\right)$$

$$\sigma_\theta^{\cancel{reg}\,l}\Big|_{r=a} = \frac{0.15^2\times 140\times10^6}{0.25^2-0.15^2}\left(1+\left(\frac{0.25}{0.15}\right)^2\right) = 297.5\ MPa$$

$$\sigma_\theta^{\cancel{reg}\,l}\Big|_{r=b} = \frac{0.15^2\times 140\times10^6}{0.25^2-0.15^2}\left(1+\left(\frac{0.25}{0.2}\right)^2\right) = 201.8\ MPa$$

$$\sigma_\theta^{\cancel{reg}\,l}\Big|_{r=c} = \frac{0.15^2\times 140\times10^6}{0.25^2-0.15^2}\,(1+1) = 157.5\ MPa$$

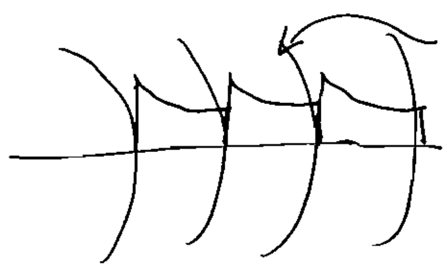
Conclusions

(1) The maximum σ_θ reduce to 257.8 MPa from (297.5 MPa)

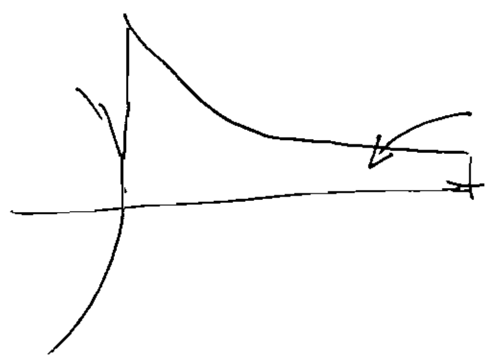
$$\frac{297.5-257.8}{297.5} \cong 14\% \quad \text{reduction}$$

$\boxed{\text{efficient use of materials}}$

(2) Multiple jacket cylinder



almost uniform stress distribution



materials here is only under lower stress state, they are not fully utilized.

(3) There are ~~other~~ pre-stress designed in other structural members, such as beams & plates.

structure members are constructed by using