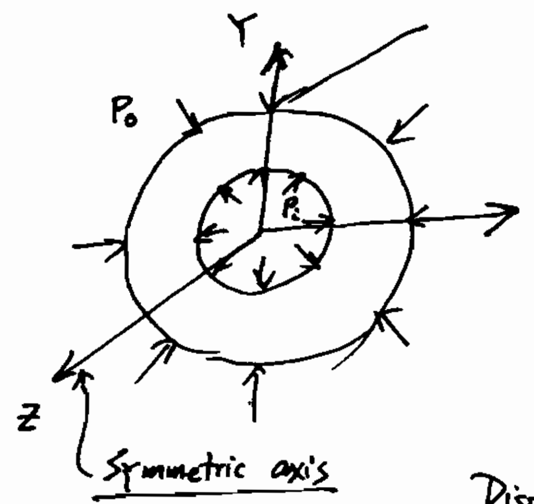
Lecture 14 Axisymmetric Loaded Member: The Thick Cylinder

Axi-symmetric problem : both deformation and external load distribution are symmetric with respect to z-axis



Applications : pipes, presure vessels dome, tunel structures

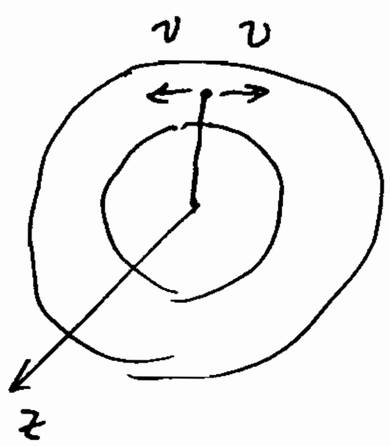
Consequences

displacement fields, stress/strain fields are independent from θ

Displacements

$$\begin{cases} u_r = u(r) \leftarrow \text{radial displacement} \\ u_\theta = v(r) \longleftarrow \text{tangential displacement} \end{cases}$$

lets examine $u_\theta = v(r)$

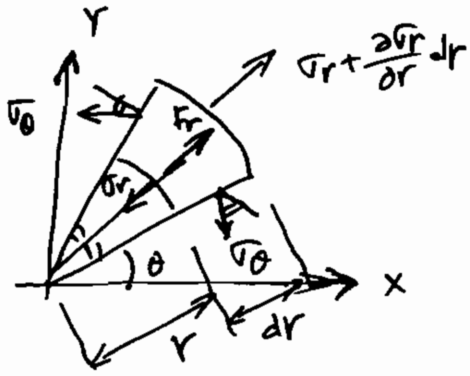


which v ? $\qquad v = -v \Rightarrow \boxed{v \equiv 0}$

Remark In 3D, the axi-symmetric problem is described as follows

$$u_r = u(r, z)$$
$$u_\theta = v \equiv 0 \qquad \leftarrow \text{In literature}$$
$$u_z = w(r, z)$$

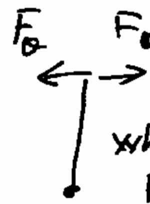
§14-1 Equilibrium equation



$\frac{d\theta}{2}$

$$\begin{cases} \sigma_r = \sigma_r(r) \\ \sigma_\theta = \sigma_\theta(r) \\ \tau_{r\theta} = 0 \end{cases}$$

$\boxed{F_\theta \equiv 0}$



which F_θ ?

$\Sigma F_r = 0$

$$\left(\sigma_r + \frac{\partial \sigma_r}{\partial r} dr\right)(r+dr)d\theta - \sigma_r r d\theta$$

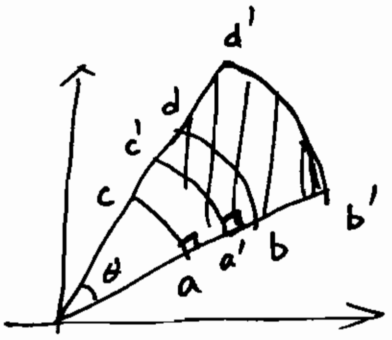
$$- 2\sigma_\theta \sin\frac{d\theta}{2} \cdot dr + F_r (r d\theta) dr = 0$$

$$\sigma_r dr d\theta + \frac{\partial \sigma_r}{\partial r} dr (r+dr) d\theta$$

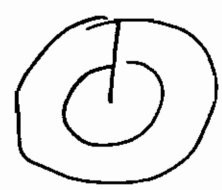
$$- \sigma_\theta d\theta dr + F_r r d\theta dr = 0$$

$$\Rightarrow \boxed{\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0}$$

Geometric equations (strain)



No shear strain since the right angle remains 90°



a : $u_r(r)$ a' : $r + u(r)$

b : $u(r+dr)$ b' : $r + dr + u(r+dr)$

$$\varepsilon_r = \frac{\overline{a'b'} - \overline{ab}}{\overline{ab}} = \frac{[dr + \overbrace{u(r+dr) - u(r)}^{\Delta u \;\leftarrow\; \text{elongation}}] - dr}{dr} = \frac{du}{dr}$$

$$\varepsilon_\theta = \frac{\widehat{a'c'} - \widehat{ac}}{\widehat{ac}} = \frac{(r+u(r))\,d\theta - r\,d\theta}{r\,d\theta} = \frac{u}{r}$$

$$\varepsilon_{r\theta} = 0$$

Assume that both ends of the cylinder are free, i.e. $\sigma_z = 0$.

By generalized Hooke's law (plane stress)

$$\varepsilon_r = \frac{du}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\varepsilon_\theta = \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

$$\boxed{\gamma_{r\theta} = \frac{1}{G}\sigma_{r\theta} \equiv 0}$$

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) \quad \leftarrow \boxed{\text{plane stress}}$$

$\sigma_z = 0$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

$$\tau_{r\theta} = 0$$

Recall the equilibrium equations in polar coordinates

$$\begin{cases} \dfrac{\partial \sigma_r}{\partial r} + \dfrac{1}{r}\cancel{\dfrac{\partial \tau_{r\theta}}{\partial \theta}} + \dfrac{\sigma_r - \sigma_\theta}{r} + F_r = 0 \\ \cancel{\dfrac{1}{r}\dfrac{\partial \sigma_\theta}{\partial \theta}} + \cancel{\dfrac{\partial \tau_{r\theta}}{\partial r}} + \cancel{\dfrac{2\tau_{r\theta}}{r}} + F_\theta = 0 \end{cases}$$

$$\Rightarrow \quad \boxed{\begin{aligned} &\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0 \\ &F_\theta = 0 \end{aligned}}$$

In absent of body force, we have

$$\boxed{\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0}$$

$$\Rightarrow \quad \cancel{\frac{E}{1-\nu^2}}\left(\frac{d^2u}{dr^2} + \frac{\nu}{r}\frac{du}{dr} - \frac{\nu}{r^2}u\right)$$

$$+ \frac{1}{r}\cancel{\frac{E}{1-\nu^2}}\left((1-\nu)\frac{du}{dr} + \frac{(\nu-1)}{r}u\right) = 0$$

$$\boxed{\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0} \quad \Leftarrow \quad \boxed{\text{Governing equation}}$$

$$\Rightarrow \quad \boxed{\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}(ru)\right\} = 0} \quad \leftarrow \text{layer form}$$

$$\frac{d}{dr}\left[\frac{1}{r}\left(r\frac{du}{dr} + u\right)\right] = \frac{d}{dr}\left[\frac{du}{dr} + \frac{u}{r}\right] = \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

step1: $$\frac{1}{r}\frac{d}{dr}(ru) = c_1'$$

$$\frac{d}{dr}(ru) = c_1' r$$

step2: $$ru = \frac{c_1' r^2}{2} + c_2'$$

$$u = \frac{c_1' r}{2} + \frac{c_2'}{r} = c_1 r + \frac{c_2}{r}$$

Then

$$\varepsilon_r = \frac{du}{dr} = c_1 - \frac{c_2}{r^2}$$

$$\varepsilon_\theta = \frac{u}{r} = c_1 + \frac{c_2}{r^2}$$

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) = \frac{E}{1-\nu^2}\left(c_1(1+\nu) - c_2\frac{(1-\nu)}{r^2}\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right) = \frac{E}{1-\nu^2}\left(c_1(1+\nu) + c_2\frac{(1-\nu)}{r^2}\right)$$

$$\sigma_r + \sigma_\theta = \frac{2Ec_1}{1-\nu^2}(1+\nu) = \frac{2Ec_1}{1-\nu} = \text{Const.}$$

$$\begin{cases} \varepsilon_z = -\frac{\nu}{E}(\sigma_r + \sigma_\theta) = -\frac{2\nu c_1}{1-\nu} = \text{Const.} & \leftarrow \boxed{\text{plane stress}} \\ \sigma_z = -\nu(\sigma_r + \sigma_\theta) = \text{Const.} & \leftarrow \text{plane strain} \end{cases}$$

Consider stress boundary conditions: To determine c_1 & c_2.

$$\sigma_r\big|_{r=a} = -P_i \quad \leftarrow \text{inner radius}$$

$$\sigma_r\big|_{r=b} = -P_o \quad \leftarrow \text{outer radius}$$

$$\frac{E}{1-\nu^2}\begin{bmatrix} (1+\nu), & -\frac{(1-\nu)}{a^2} \\ (1+\nu), & -\frac{(1-\nu)}{b^2} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -P_i \\ -P_o \end{bmatrix}$$

Solve for C_1 & C_2

$$C_1 = \left(\frac{1-\nu}{E}\right)\frac{a^2P_i - b^2P_o}{b^2-a^2}$$

$$C_2 = \left(\frac{1+\nu}{E}\right)\frac{a^2b^2(P_i-P_o)}{b^2-a^2}$$

Remarks

has nothing to do with ν & E

So they are valid for both

Finally,

$$\sigma_r = \frac{a^2P_i - b^2P_o}{b^2-a^2} - \frac{(P_i-P_o)a^2b^2}{(b^2-a^2)r^2}$$

$$\sigma_\theta = \frac{a^2P_i - b^2P_o}{b^2-a^2} + \frac{(P_i-P_o)a^2b^2}{(b^2-a^2)r^2}$$

$a \le r \le b$

$$u = \left(\frac{1-\nu}{E}\right)\frac{(a^2P_i - b^2P_o)r}{b^2-a^2} + \left(\frac{1+\nu}{E}\right)\frac{(P_i-P_o)a^2b^2}{(b^2-a^2)r}$$ ← plane stress

$$u = \frac{(1+\nu)(1-2\nu)}{E}\frac{(a^2P_i - b^2P_o)r}{b^2-a^2} + \frac{1+\nu}{E}\frac{(P_i-P_o)a^2b^2}{(b^2-a^2)r}$$ ← plane strain

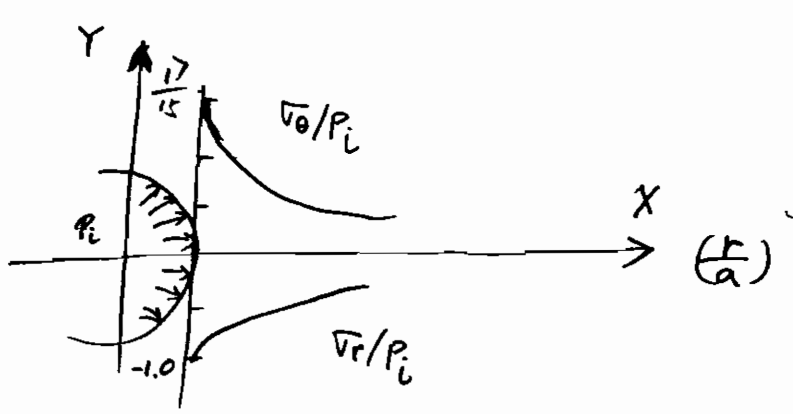
(1) $P_o = 0$,

$$\sigma_r = \frac{a^2P_i}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right) < 0, \quad \text{Note that } \frac{b^2}{r^2} > 1, \quad a \le r \le b$$

$$\sigma_\theta = \frac{a^2P_i}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right) > 0, \quad \text{When } b\to\infty, \quad \sigma_\theta = P_i \text{ at } r=a,$$

This is the best you can do

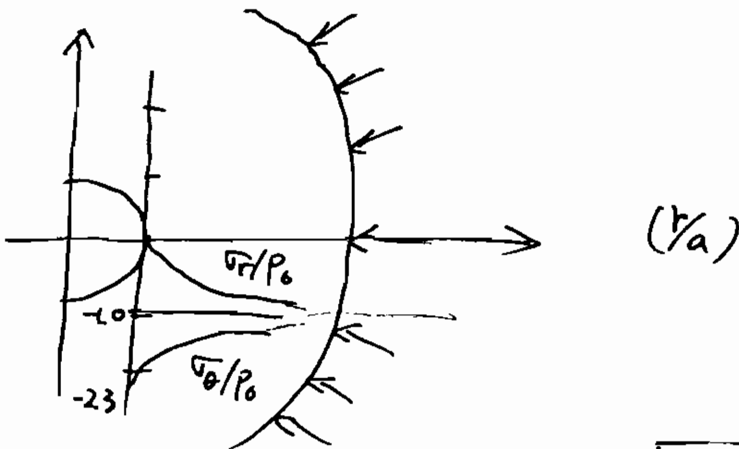
$b/a = 4$



(2) $p_i = 0$

$$\sigma_r = -\frac{p_o b^2}{b^2-a^2}\left(1-\frac{a^2}{r^2}\right) < 0, \qquad \text{Note that} \quad 0 \le \frac{a^2}{r^2} \le 1, \quad a \le r \le b$$

$$\sigma_\theta = -\frac{p_o b^2}{b^2-a^2}\left(1+\frac{a^2}{r^2}\right) < 0$$

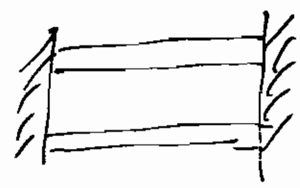


plane strain

$$\boxed{\sigma_z = \frac{-2\nu\,(a^2 p_i - b^2 p_o)}{b^2-a^2}}$$

§ 16.2 Application of Failure theories

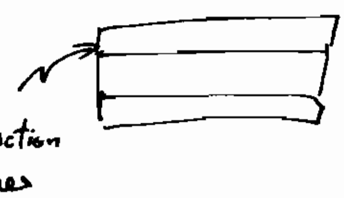
(*) Assume that this is plane strain state. Therefore



$$\sigma_z = -\nu(\sigma_{rr}+\sigma_{\theta\theta}) = c.$$

(*) Assume that the ends of the cylinder are open and unconstrained, $\sigma_z = 0$.

$$\int_a^b \sigma_z \, 2\pi r\, dr = \pi c\,(b^2-a^2) = 0$$



$$c = 0$$

$$\Rightarrow \boxed{\sigma_z = 0}$$

Tri-axial stress state

plane stress $\rightarrow$ plane strain $E \rightarrow \dfrac{E}{1-\nu^2}$, $\nu \rightarrow \dfrac{\nu}{1-\nu}$

$$\left\{\begin{array}{l} 1-\nu = 1 - \dfrac{\nu}{1-\nu} = \dfrac{1-2\nu}{1-\nu} \\ \dfrac{1}{E} = \dfrac{1-\nu^2}{E} \end{array}\right\} \quad \frac{1-\nu}{E} = \frac{(1-\nu^2)\,\frac{1-2\nu}{1-\nu}}{E} = \frac{(1+\nu)(1-2\nu)}{E}$$

$$\left\{\begin{array}{l} 1+\nu = 1 + \dfrac{\nu}{1-\nu} = \dfrac{1}{1-\nu} \\ \dfrac{1}{E} = \dfrac{1-\nu^2}{E} \end{array}\right\} \quad \frac{1+\nu}{E} = \frac{(1-\nu^2)\,\frac{1}{1-\nu}}{E} = \frac{1+\nu}{E}$$

§16.2 Application of Failure theories

(★) plane strain $\varepsilon_{zz} = 0$

$$\sigma_z = -\nu\,(\sigma_r + \sigma_\theta) = -2\nu\,\frac{a^2 p_i - b^2 p_o}{b^2 - a^2} = \text{Const.}$$

(★) plane stress $\sigma_{zz} = 0$

Consider plane stress

(1) Maximum shear stress theory

$\{ \sigma_\theta, \ \sigma_z, \ \sigma_r \}$

At $r=a$ $\quad \{ \sigma_\theta, \ \sigma_z, \ \sigma_r \}$ $\quad \boxed{p_o = 0}$

$$\{ \sigma_1 > 0 \ \ \sigma_2 = 0 \ \ \sigma_3 < 0 \}$$

$$\tau_{max} = \frac{1}{2}(\sigma_\theta - \sigma_r) = \tau_{YP}$$

$$\boxed{\frac{3}{4G}\tau_{oct}^2 = U_o^d}$$

(2) Maximum distortion energy theory

$$U_o^d = \frac{1}{12G}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right] = \frac{1}{6G}\sigma_{YP}^2$$

$$(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2\sigma_{YP}^2$$

$$\frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2} \triangleq \tau_{oct} = \frac{\sqrt{2}}{3}\sigma_{YP}$$

$$\sigma_{YP} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}$$

τ_{oct}

$$= \frac{1}{\sqrt{2}}\sqrt{(\sigma_\theta)^2 + \sigma_r^2 + (\sigma_\theta - \sigma_r)^2}$$

Octahedral stress

$$\boxed{\sigma_{YP} = \sqrt{\sigma_\theta^2 + \sigma_r^2 - \sigma_r\sigma_\theta}}$$

(3) Example 8.2 $\quad \tau_{YP} = \sigma_{YP}/2 = 170$ MPa

$a = 0.1$ m $\quad b = 0.15$ m $\quad \frac{b}{a} = \frac{3}{2} = 1.5$

$p_i = 4p_o$

Find the allowable p_i ?

Solution :

$$\sigma_\theta = \frac{P_i(a^2+b^2) - \cancelto{0}{2P_o}\, b^2}{b^2-a^2} = 1.7 P_i\,, \qquad \sigma_r = -P_i$$

(1) Maximum shear stress criterion

$$\frac{\sigma_\theta - \sigma_r}{2} = \frac{1.7P_i - (-P_i)}{2} = 1.35 P_i = \tau_{YP} = 170 \text{ MPa}$$

$$\Rightarrow \quad P_i = \frac{170}{1.35} = \boxed{125.9 \text{ MPa}}$$

Compare with $\sigma_{YP} = 340$ MPa

Conventional wisdom

(2) Maximum distortion energy criterion

$$\sigma_{YP} = \sqrt{\sigma_\theta^2 + \sigma_r^2 - \sigma_\theta\sigma_r}$$

$$= \sqrt{(1.7P_i)^2 + P_i^2 + 1.7P_i^2} = P_i\sqrt{1.7^2+1+1.7}$$

$$= 2.364 P_i = 340 \text{ MPa}$$

$$\boxed{P_i = 143.8 \text{ MPa}}$$

If $b \to \infty$, $\sigma_\theta = P_i$, $\sigma_r = -P_i$, $\sigma_z = 0$

The best you can do

$$\frac{\sigma_\theta - \sigma_r}{2} = \frac{P_i - (-P_i)}{2} = P_i = \boxed{170 \text{ MPa}}$$

$\sigma_{YP} = 340$ MPa