Solution to Practice Final ~~Exam~~ Examination

Problem 1

[Solution] $$\nabla^2\nabla^2 w = \frac{P(x,y)}{D}$$

Assume that

$$W(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a}$$

for a simply supported square plate.

We find that

$$a_{mn} = \frac{P_{mn}}{\pi^4 D \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2} = \frac{P_{mn}\, a^4}{\pi^4 D\, [m^2+n^2]^2}$$

where

$$P_{mn} = \frac{4}{a^2}\int_0^a\int_0^a \frac{P_0}{a}(a-x) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a}\, dx\, dy$$

Consider

$$\int_0^a (a-x)\sin\frac{m\pi x}{a}\,dx = -\frac{a}{m\pi}\cos\frac{m\pi x}{a}(a-x)\Big|_0^a + \frac{a}{m\pi}\int_0^a \cos\frac{m\pi x}{a}(-1)\,dx$$

$$= \frac{a^2}{m\pi}$$

$$\int_0^a \sin\frac{n\pi y}{a}\,dx = -\frac{a}{n\pi}\cos\frac{n\pi y}{a}\Big|_0^a = \frac{a}{n\pi}\left(1-(-1)^n\right)$$

$$P_{mn} = \left(\frac{4P_0}{\cancel{a^3}}\right)\cdot\left(\frac{\cancel{a^2}}{m\pi}\right)\left(\frac{\cancel{a}}{n\pi}\right)\left(1-(-1)^n\right) = \frac{4P_0}{m\cdot n\pi^2}\left(1-(-1)^n\right)$$

$$a_{mn} = \left(\frac{4P_0 a^4}{\pi^6 D}\right) \frac{(1-(-1)^n)}{mn\,(m^2+n^2)^2}$$

$\blacksquare$

Problem 2

$$P_2 = P, \qquad P_1 = -P$$

[Solution]

(a) $$v(x) = \frac{P\beta}{2k} f_1(\beta x) - \frac{P\beta}{2k} f_1[\beta(x+\delta)]$$

(b) $$v(x) = -\frac{(P\delta)\beta}{2k} \frac{f_1[\beta(x+\delta)] - f_1(\beta x)}{\delta}$$

As $\delta \to 0$, $\qquad P\delta = M_0$

$$\lim_{\delta\to 0} v(x) = -\frac{M_0\beta}{2k} \lim_{\delta\to 0} \frac{f_1[\beta(x+\delta)] - f_1(\beta x)}{\delta}$$

$$= -\frac{M_0\beta}{2k} \frac{d}{dx} \boxed{f_1(\beta x)}$$

$$\frac{d}{dx} f_1(\beta x) = -2\beta f_2(\beta x)$$

$$v(x) = \left(-\frac{M_0\beta}{2k}\right)\left(-2\beta f_2(\beta x)\right) = \frac{M_0\beta^2}{k} f_2(\beta x)$$

$$v'(x) = \frac{M_0\beta^2}{k} f_2'(\beta x) = \frac{M_0\beta^3}{k} f_3(\beta x)$$

$$M(x) = EI v''(x)$$

$$= EI \frac{M_0\beta^3}{k} f_3'(\beta x) = EI \frac{M_0\beta^3}{k} (-2\beta f_4(\beta x))$$

$$\boxed{M(x) = -\frac{EI M_0 \beta^4}{k} f_4(\beta x)}$$

Problem 3

$a = 100$ mm, $b = 150$ mm, $c = 200$ mm, $P_e = 200$ MPa

[Solution]

step1 — find interfacial pressure

$$p = \frac{E\delta}{b} \frac{(b^2-a^2)(c^2-b^2)}{2b^2(c^2-a^2)}$$

$$= \left(\frac{72\times10^9 \times 0.125\times10^{-3}}{150\times10^{-3}}\right)\times\left(\frac{1}{2}\right)\frac{(150^2-100^2)(200^2-150^2)}{150^2(200^2-100^2)}$$

$$= 9.72\dot{2} \text{ MPa}$$

step 2

$$\sigma_{\theta\theta} = \frac{a^2 P_e}{c^2-a^2}\left(1+\frac{c^2}{r^2}\right) + \begin{cases} -\dfrac{b^2 p}{b^2-a^2}\left(1+\dfrac{a^2}{r^2}\right), & a \le r \le b \\ \dfrac{b^2 p}{c^2-b^2}\left(1+\dfrac{c^2}{r^2}\right), & b \le r \le c \end{cases}$$

At $r = b^-$

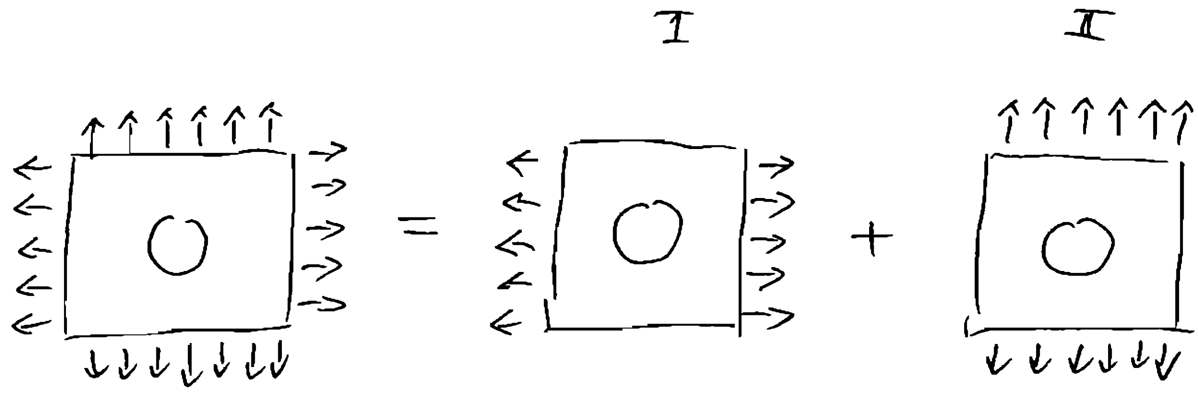
$$\sigma_{\theta\theta} = \frac{100^2\times200\times10^6}{200^2-100^2}\left(1+\frac{200^2}{150^2}\right) - \frac{150^2\times9.722\times10^6}{150^2-100^2}\left(1+\left(\frac{2}{3}\right)^2\right)$$

$$= 185.185 \text{ MPa} - 25.272 \text{ MPa} = 159.9 \text{ MPa}$$

At $r = b^+$

$$\sigma_{\theta\theta} = 185.185 \text{ MPa} + \frac{150^2\times9.722\times10^6\left(1+\left(\frac{4}{3}\right)^2\right)}{200^2-150^2} = 219.9 \text{ MPa}$$

Problem 4 [Solution]



[Solution]

$$\sigma_r^{I} = \frac{1}{2}\sigma_0 \left[\left(1 - \frac{a^2}{r^2}\right) + \left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta \right]$$

$$\sigma_\theta^{I} = \frac{1}{2}\sigma_0 \left[\left(1 + \frac{a^2}{r^2}\right) - \left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta \right]$$

$$\sigma_r^{II} = \frac{1}{2}(-3\sigma_0) \left[\left(1 - \frac{a^2}{r^2}\right) + \left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\phi \right]$$

$$\sigma_\theta^{II} = \frac{1}{2}(-3\sigma_0) \left[\left(1 + \frac{a^2}{r^2}\right) - \left(1 + \frac{3a^4}{r^4}\right)\cos 2\phi \right]$$

$$\phi = \frac{\pi}{2} + \theta \; ; \qquad \cos 2\phi = -\cos 2\theta$$

At $r = a$ $\quad \sigma_r \equiv 0$

$$\sigma_\theta^{I} = \frac{\sigma_0}{2}\left[2 - 4\cos 2\theta\right] = \sigma_0\left[1 - 2\cos 2\theta\right]$$

$$\sigma_\theta^{II} = -\frac{3\sigma_0}{2}\left[2 + 4\cos 2\theta\right]$$

$$\sigma_\theta = \sigma_\theta^{I} + \sigma_\theta^{II} = \sigma_0 [1 - 2\cos 2\theta] - \frac{3\sigma_0}{2} \times 2 [1 + 2\cos 2\theta]$$

$$= -2\sigma_0 - 8\sigma_0 \cos 2\theta$$

At $\theta = 0$

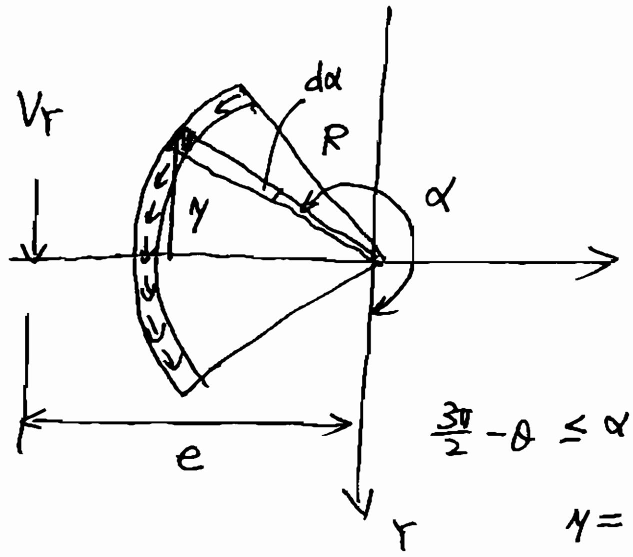
$$\sigma_{\theta\theta} = \sigma_{rr} = -10\sigma_0 \quad \Leftarrow \quad \boxed{|\sigma_{rr}| = \sigma_{rr\,max}}$$

At $\theta = \frac{\pi}{2}$ $\quad \cos 2\theta = -1$

$$\sigma_{\theta\theta} = \sigma_{xx} = 6\sigma_0 = (\sigma_{xx})_{max}$$

Problem 5

[Solution]



$$V_Y \cdot e = \int_{\frac{3\pi}{2}-\theta}^{\frac{3\pi}{2}+\theta} (R\,\tau_{x\alpha})\, dA$$

$$\frac{3\pi}{2}-\theta \le \alpha \le \frac{3\pi}{2}+\theta$$

$$dA = Rt\, d\alpha$$

$$y = R\cos\alpha$$

$$\tau_{x\alpha} = -\frac{V_Y\, Q(y)}{t\, I_z}$$

$$I_z = \int_{\frac{3\pi}{2}-\theta}^{\frac{3\pi}{2}+\theta} R^2\cos^2\alpha\,(Rt\, d\alpha) = \frac{R^3 t}{2}\int_{\frac{3\pi}{2}-\theta}^{\frac{3\pi}{2}+\theta}(1-\cos 2\alpha)\,d\alpha$$

$$= \frac{R^3 t}{2}\left[2\theta - 2\sin 2\theta\right] = \frac{R^3 t}{2}\cdot 2\left[\theta - \sin 2\theta\right]$$

$$= R^3 t\,(\theta - \sin 2\theta)$$

$$Q(y) = \int_{\frac{3\pi}{2}-\theta}^{\alpha} y\, dA = \int_{\frac{3\pi}{2}-\theta}^{\alpha} R\cos\alpha\,(Rt\, d\alpha)$$

$$= R^2 t \sin\alpha \Big|_{\frac{3\pi}{2}-\theta}^{\alpha} = R^2 t\left(\sin\alpha - \sin\left(\frac{3\pi}{2}-\theta\right)\right)$$

$$= R^2 t\,(\sin\alpha + \cos\theta)$$

$$e \cdot \cancel{V_Y} = \int_{\frac{3\pi}{2}-\theta}^{\frac{3\pi}{2}+\theta} R\cdot\left(-\frac{\cancel{V_Y}\cdot \cancel{R^2 t}\,(\sin\alpha + \cos\theta)}{\cancel{t}\cdot \cancel{R^3 t}\,(\theta - \sin 2\theta)}\right)\cancel{Rt}\, d\alpha$$

$$e = -\int_{\frac{3\pi}{2}-\theta}^{\frac{3\pi}{2}+\theta} R \,\frac{(\sin\alpha + \cos\theta)}{(\theta - \sin 2\theta)}\, d\alpha$$

$$= -\left(\frac{R}{\theta - \sin 2\theta}\right)\left[-\cos\alpha + \cos\theta\cdot\alpha\right]\Big|_{\frac{3\pi}{2}-\theta}^{\frac{3\pi}{2}+\theta}$$

$$= \frac{R}{\theta - \sin 2\theta}\cdot\left[(2\sin\theta) - 2\cos\theta\cdot\theta\right] = \frac{2R}{\theta - \sin 2\theta}\cdot\left[\sin\theta - \cos\theta\cdot\theta\right]$$

$$\boxed{e = \frac{2R}{(\theta - \sin 2\theta)}\left[\sin\theta - \theta\cdot\cos\theta\right]}$$