Problem I

a) $\delta_{ij} c_{jn} = c_{i1}\delta_{1n} + c_{i2}\delta_{2n} + c_{i3}\delta_{3n} = c_{in}$

b) $a_{ij} \delta_{jn} = a_{1j}\delta_{1n} + a_{2j}\delta_{2n} + a_{3j}\delta_{3n} = a_{nj}$

c) $\delta_{ij}\delta_{jn}\delta_{ni} = \delta_{ij}\delta_{ji} = \delta_{ii} = \delta_{nn} = \delta_{jj} = 1+1+1 = 3$

e) $\delta_{ij}\delta_{ij} = \delta_{ii} = \delta_{jj} = \delta_{11}+\delta_{22}+\delta_{33} = 1+1+1 = 3$

f) $\delta_{ii} = \delta_{11}+\delta_{22}+\delta_{33} = 1+1+1 = 3$

g) $A_i B_j \delta_{ij} - B_m A_n \delta_{mn} = A_i B_i - B_m A_m = A_1B_1 + A_2B_2 + A_3B_3 - (B_1A_1 + B_2A_2 + B_3A_3) = 0$

Problem II

$l_1 = \cos(e'_1, e_1) = e'_1 \cdot e_1 = Q_{11}$ $m_1 = \cos(e'_1, e_2) = e'_1 \cdot e_2 = Q_{12}$ $n_1 = \cos(e'_1, e_3) = e'_1 \cdot e_3 = Q_{13}$

$l_2 = \cos(e'_2, e_1) = e'_2 \cdot e_1 = Q_{21}$ $m_2 = \cos(e'_2, e_2) = e'_2 \cdot e_2 = Q_{22}$ $n_2 = \cos(e'_2, e_3) = e'_2 \cdot e_3 = Q_{23}$

$l_3 = \cos(e'_3, e_1) = e'_3 \cdot e_1 = Q_{31}$ $m_3 = \cos(e'_3, e_2) = e'_3 \cdot e_2 = Q_{32}$ $n_3 = \cos(e'_3, e_3) = e'_3 \cdot e_3 = Q_{33}$

This terms can be arrange in the matrix $[Q_{ik}]$. And the next operation can be done to prove $l_i^2 + m_i^2 + n_i^2 = 1$:

$$Q_{ik}^{-1} = Q_{kj}^T$$
$$Q_{ik} Q_{kj}^{-1} = Q_{ik} Q_{kj}^T$$
$$\delta_{ij} = Q_{ik} Q_{kj}^T$$

Where $[Q_{ik}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$ and $[Q_{kj}]^T = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix}$

Then $[Q_{ik}][Q_{kj}]^T =$

$$\begin{bmatrix} Q_{11}^2 + Q_{12}^2 + Q_{13}^2 & Q_{11}Q_{21} + Q_{12}Q_{22} + Q_{13}Q_{23} & Q_{11}Q_{31} + Q_{12}Q_{32} + Q_{13}Q_{33} \\ Q_{11}Q_{21} + Q_{12}Q_{22} + Q_{13}Q_{23} & Q_{21}^2 + Q_{22}^2 + Q_{23}^2 & Q_{21}Q_{31} + Q_{22}Q_{32} + Q_{23}Q_{33} \\ Q_{11}Q_{31} + Q_{12}Q_{32} + Q_{13}Q_{33} & Q_{21}Q_{31} + Q_{22}Q_{32} + Q_{23}Q_{33} & Q_{31}^2 + Q_{32}^2 + Q_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Replacing with l_i, m_i and n_i:

$$\begin{bmatrix} l_1^2 + m_1^2 + n_1^2 & l_1 l_2 + m_1 m_2 + n_1 n_2 & l_1 l_3 + m_1 m_3 + n_1 n_3 \\ l_1 l_2 + m_1 m_2 + n_1 n_2 & l_2^2 + m_2^2 + n_2^2 & l_2 l_3 + m_2 m_3 + n_2 n_3 \\ l_1 l_3 + m_1 m_3 + n_1 n_3 & l_2 l_3 + m_2 m_3 + n_2 n_3 & l_3^2 + m_3^2 + n_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem III

Show $[B_{ij}]$ is orthogonal $\rightarrow [B_{ik}][B_{kj}^T] = [B_{ik}^T][B_{kj}] = [\delta_{ij}]$

$$[B_{ij}] = \begin{bmatrix} 12/25 & 3/5 & -16/25 \\ -9/25 & 4/5 & 12/25 \\ 4/5 & 0 & 3/5 \end{bmatrix}$$

$$[B_{kj}^T] = \begin{bmatrix} 12/25 & -9/25 & 4/5 \\ 3/5 & 4/5 & 0 \\ -16/25 & 12/25 & 3/5 \end{bmatrix}$$

$[B_{ik}][B_{kj}^T] =$

$$\begin{bmatrix} (12/25)^2 + (3/5)^2 + (-16/25)^2 = 1 & [illegible] = 0 & [illegible] = 0 \\ (12/25)(-9/25) + (3/5)(4/5) + [illegible] = 0 & [illegible] = 1 & [illegible] = 0 \\ (12/25)(4/5) + (0)(3/5) + [illegible] = 0 & [illegible] = 0 & [illegible] = 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$[B_{ik}^T][B_{kj}]$

$$\begin{bmatrix} [illegible] (4/5)^2 = 1 & [illegible] = 0 & [illegible] = 0 \\ [illegible] = 0 & (3/5)^2 + (4/5)^2 [illegible] = 1 & [illegible] = 0 \\ [illegible] = 0 & [illegible] = 0 & [illegible] = 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem IV

$$a_{ij} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

a) $a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 2 + 3 = 6$

b) $a_{ij}a_{ij} = a_{ij}^2 = a_{12}^2 + a_{13}^2 + a_{21}^2 + a_{23}^2 + a_{31}^2 + a_{32}^2 + a_{11}^2 + a_{22}^2 + a_{33}^2$

$= 1^2 + 0 + 1^2 + 2^2 + 0 + 2^2 + 1^2 + 2^2 + 3^2 = 24$

c) $a_{ij}a_{ik} = a_{1j}a_{1k} + a_{2j}a_{2k} + a_{3j}a_{3k}$

when $j=1$, $k=1$: $a_{ij}a_{ik} = a_{11}a_{11} + \dots$

$= a_{11}^2 + a_{21}^2 + a_{31}^2$

$= 1^2 + 1^2 + 0 = 2$

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when $j=1$, $k=2$: $a_{ij}a_{ik} = a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32}$

$= (1)(1) + (1)(2) + \dots = 3$

Problem V

$[T'_{ij}] = [Q_{ik}^T][T_{kl}][Q_{lj}]$

$$[Q_{ik}]^T = [Q_{ij}]^T = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \dots = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_{kl} = T_{ij} = \begin{bmatrix} 10 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Then

$$[T'_{ij}] = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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With calculator:

$$[T'_{ij}] = \begin{bmatrix} \sqrt{3}/2 + 35/4 & -\sqrt{3}/4 - 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/4 - 1/2 & 55/4 - \sqrt{3}/2 & 5/2 \\ \sqrt{3}/2 & 5/2 & 1 \end{bmatrix} = \begin{bmatrix} 7.12 & -0.93 & 0.87 \\ -0.93 & 13.88 & 2.50 \\ 0.87 & 2.50 & 1.00 \end{bmatrix}$$

Problem VI

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \text{grad})\,\underline{v} = \underline{f} - \frac{1}{\rho}\,\text{grad}\,p + \nu \Delta v$$

① $\underline{V} = V_i \underline{e}_i$

② $\text{grad} = \frac{\partial}{\partial x_i} e_i$

③ $\Delta = \nabla^2 = \frac{\partial^2}{\partial x_i \partial x_i} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$

④ $\frac{\partial \underline{v}}{\partial t} = \underline{\dot{v}} = \dot{V}_i \underline{e}_i = \frac{\partial V_i}{\partial t} \underline{e}_i$

⑤ $\underline{V} \cdot \text{grad} = (V_i \underline{e}_i) \cdot \left(\frac{\partial \underline{e}_j}{\partial x_j}\right) = V_i \frac{\partial}{\partial x_j} \underline{e}_i \cdot \underline{e}_j = V_i \frac{\partial \delta_{ij}}{\partial x_j} = V_i \frac{\partial}{\partial x_i} =$

⑥ $(\underline{V} \cdot \text{grad})\,V = V_i \frac{\partial}{\partial x_i} (V_j \underline{e}_j) = V_i \frac{\partial V_j}{\partial x_i} \underline{e}_j = V_j \frac{\partial V_i}{\partial x_j} \underline{e}_j$

⑦ $\underline{f} = f_i \underline{e}_i$

⑧ $\frac{1}{\rho}\left(\frac{\partial}{\partial x_i} \underline{e}_i\right) p = \frac{1}{\rho} \frac{\partial p}{\partial x_i} \underline{e}_i$

⑨ $\nu \frac{\partial^2}{\partial x_j \partial x_j} (V_i \underline{e}_i) = \nu \frac{\partial^2 V_i}{\partial x_j \partial x_j} \underline{e}_i$

Taking into account equations ① trough ⑨ the expression can be written like this:

$$\boxed{\frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} = f_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 V_i}{\partial x_j \partial x_j} \quad i = 1, 2, 3}$$

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