CHOU Ci ESTHER CHAU
1544-0499

CEE 131 : Problem Set #7

Ads 7-1 $\Phi = -\dfrac{M(\sin 2\theta - 2\theta\cos 2\alpha)}{2(\sin 2\alpha - 2\alpha\cos 2\alpha)}$

a) $\nabla^4\Phi = 0$

$\nabla^2\Phi = \left(\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right)\left[-\dfrac{M(\sin 2\theta - 2\theta\cos 2\alpha)}{2(\sin 2\alpha - 2\alpha\cos 2\alpha)}\right]$

$= -\dfrac{M}{2r^2(\sin 2\alpha - 2\alpha\cos 2\alpha)}\dfrac{d^2}{d\theta^2}(\sin 2\theta - 2\theta\cos 2\alpha)$

$= -\dfrac{M}{2r^2(\sin 2\alpha - 2\alpha\cos 2\alpha)}\dfrac{d}{d\theta}(2\cos 2\theta - 2\cos 2\alpha)$

$= -\dfrac{M}{2r^2(\sin 2\alpha - 2\alpha\cos 2\alpha)}(-4\sin 2\theta)$

$= \dfrac{2M\sin 2\theta}{r^2(\sin 2\alpha - 2\alpha\cos 2\alpha)}$

$\nabla^2\nabla^2\Phi = \left(\dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr} + \dfrac{1}{r^2}\dfrac{d^2}{d\theta^2}\right)\left[\dfrac{2M\sin 2\theta}{r^2(\sin 2\alpha - 2\alpha\cos 2\alpha)}\right]$

$= \dfrac{2M}{\sin 2\alpha - 2\alpha\cos 2\alpha}\left(\dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr} + \dfrac{1}{r^2}\dfrac{d^2}{d\theta^2}\right)\left[\dfrac{\sin 2\theta}{r^2}\right]$

$= \dfrac{2M}{\sin 2\alpha - 2\alpha\cos 2\alpha}\left[6\dfrac{\sin 2\theta}{r^4} - 2\dfrac{\sin 2\theta}{r^4} - \dfrac{1}{r^4}(4\sin 2\theta)\right]$

$= \dfrac{2M}{\sin 2\alpha - 2\alpha\cos 2\alpha}(0)$

$= 0$

$\nabla^4\Phi = 0$

b) $\sigma_r = \left(\dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right)\Phi$

$= -\dfrac{1}{r^2}\dfrac{M}{2(\sin 2\alpha - 2\alpha\cos 2\alpha)}\dfrac{d}{d\theta}[2\cos 2\theta - 2\cos 2\alpha]$

$= -\dfrac{1}{r^2}\dfrac{M}{2(\sin 2\alpha - 2\alpha\cos 2\alpha)}(-4\sin 2\theta)$

$= \dfrac{2M\sin 2\theta}{r^2(\sin 2\alpha - 2\alpha\cos 2\alpha)}$

$\sigma_\theta = \dfrac{\partial^2}{\partial r^2}\Phi = 0$

$\tau_{r\theta} = -\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial\Phi}{\partial\theta}\right)$

$= -\dfrac{\partial}{\partial r}\left[-\dfrac{M(\cos 2\theta - \cos 2\alpha)}{r(\sin 2\alpha - 2\alpha\cos 2\alpha)}\right]$

$= -\dfrac{M(\cos 2\theta - \cos 2\alpha)}{r^2(\sin 2\alpha - 2\alpha\cos 2\alpha)}$

c) For $\alpha = \frac{\pi}{2}$,

$\sigma_r = \dfrac{2M\sin 2\theta}{r^2(\sin\pi - 2\frac{\pi}{2}\cos\pi)} = \dfrac{2M\sin 2\theta}{r^2(\pi)} = \dfrac{2M\sin 2\theta}{\pi r^2}$

$\sigma_\theta = 0$

$\tau_{r\theta} = -\dfrac{M(\cos 2\theta - \cos\pi)}{r^2[\sin\pi - 2(\frac{\pi}{2})\cos\pi]} = -\dfrac{M(\cos 2\theta + 1)}{r^2(0 - \pi(-1))} = -\dfrac{M(\cos 2\theta + 1)}{\pi r^2}$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - 1 + \cos^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2\theta + 1 = 2\cos^2\theta$

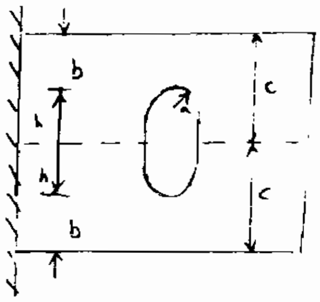
✓ (10)

$\tau_{r\theta} = -\dfrac{M(2\cos^2\theta)}{\pi r^2} = -\dfrac{2M\cos^2\theta}{\pi r^2}$

$\therefore\ \sigma_r = \dfrac{2M\sin 2\theta}{\pi r^2}$, $\sigma_\theta = 0$, $\tau_{r\theta} = -\dfrac{2M\cos^2\theta}{\pi r^2}$ can represent the stress field

CHIU CI ESTHER CHAU
1544-0499

Prob 7-3



$b = 17h$

$c = 16h$

$a = h$

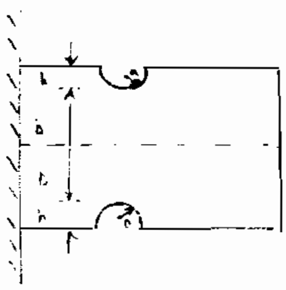
$\sqrt{\frac{b}{a}} = \sqrt{17} = 4.123$

$\sqrt{\frac{h}{a}} = 1$

axial tensile load : Scale for $\sqrt{h/a}$ is f , Curve for k is 5

From Neuber's nomograph , $k = 2.9$

Prob 7-4



$h = 3a$

$b = 15a$

$\sqrt{\frac{b}{a}} = \sqrt{15} = 3.873$

$\sqrt{\frac{h}{a}} = \sqrt{3} = 1.732$

a) Axial tensile load : Scale for $\sqrt{h/a}$ is f , Curve for k is 1

From Neuber's nomograph , $k = 3.5$

b) Bending : Scale for $\sqrt{h/a}$ is f , curve for k is 2

From Neuber's nomograph , $k = 2.9$

Prob 7-5 Verify equations

i) $\int_0^{\frac{\pi}{2}} (\sigma_r \sin\theta)\, r\, d\theta = \int_0^{\frac{\pi}{2}} \frac{2P}{\pi} \sin\theta \cos\theta \, d\theta = \frac{P}{\pi}$

ii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sigma_r \cos\theta)\, r\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2P}{\pi} \cos^2\theta \, d\theta = P$

Stress due to concentrated loads

$\sigma_r = - \frac{P\cos\theta}{r[\alpha + \frac{1}{2}\sin 2\alpha]} = - \frac{2P\cos\theta}{r\pi}$

$\sigma_\theta = 0$

$\tau_{r\theta} = 0$

CHZU LI ESIMEK UTIKU
1544 -0499

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(Prob 7/5) i) $\int_0^{\frac{\pi}{2}} (\sigma_r \sin\theta) r\, d\theta = \int_0^{\frac{\pi}{2}} \frac{2P\cos\theta}{\cancel{r}\pi} \sin\theta\, \cancel{r}\, d\theta$

$= \frac{P}{\pi} \int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta$

$= \frac{P}{\pi} \; -\frac{1}{2} \cos 2\theta \Big|_0^{\frac{\pi}{2}}$

$= -\frac{P}{2\pi} \left[(\cos\pi) - (\cos(0)) \right]$

$= -\frac{P}{2\pi} (-1-1)$

$\int_0^{\frac{\pi}{2}} (\sigma_r \sin\theta) r\, d\theta = \frac{P}{\pi}$

ii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sigma_r \cos\theta) r\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2P\cos\theta}{\cancel{r}\pi} \cos\theta \, \cancel{r}\, d\theta$

$= \frac{2P}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta$

$= \frac{2P}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\cancel{2}} (1+\cos 2\theta)\, d\theta$

$= \frac{P}{\pi} \left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{P}{\pi} \left(\frac{\pi}{2} + \frac{\pi}{2} \right)$

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sigma_r \cos\theta) r\, d\theta = P$

(1)

Prob 7-6 $\Phi = f(\theta) r^2$

Show $\Phi = r^2 (A\cos 2\theta + B\sin 2\theta + C\theta + D)$

i) $\nabla^2\nabla^2\Phi = 0$

$\nabla^2\Phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \right) (f(\theta) r^2)$

$= \frac{\partial}{\partial r} (2f(\theta) r) + \frac{1}{r} (2f(\theta)\, r) + \frac{\partial}{\partial\theta} (f'(\theta))$

$= 4f(\theta) + f''(\theta)$

$\nabla^2\nabla^2\phi = \left(\cancel{\frac{\partial^2}{\partial r^2}} + \cancel{\frac{1}{r}\frac{\partial}{\partial r}} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \right) (4f(\theta) + f''(\theta))$

$= \frac{1}{r^2} \left[4 f^2(\theta) + f^4(\theta) \right]$

$\int \frac{1}{r^2} \left[4 f^2(\theta) + f^4(\theta) \right] = 0$

$\int 4 f'(\theta) + f^3(\theta) = a_0$

$4 f(\theta) + f^2(\theta) = a_0\theta + a_1 = g(\theta)$

General solution = homogeneous + particular

$f_g(\theta) = f_h(\theta) + f_p(\theta)$

Homogeneous : $f_g(\theta) = 0 \Rightarrow 4f(\theta) + f^2(\theta) = 0$

$\hookrightarrow f_h(\theta) = A\cos 2\theta + B\sin 2\theta$

CHIN CI ESTHER CHAU
1544-0499

(Prob 7-6) Particular : $f_y(\theta) = 4(C_1\theta + C_2) + 0 = a_0\theta + a_1$

$\hookrightarrow f_p(\theta) = \frac{a_0}{4}\theta + \frac{a_1}{4}$

$\hookrightarrow C = \frac{a_0}{4}, \quad D = \frac{a_1}{4}$

$f_y(\theta) = f_h(\theta) + f_p(\theta) = A\cos 2\theta + B\sin 2\theta + C\theta + D$

$\Phi = r^2 f(\theta) = r^2(A\cos 2\theta + B\sin 2\theta + C\theta + D)$

ii) Boundary condition : When $\theta = \alpha$, traction free

$\sigma_\theta |_{\theta=\alpha} = 0$

$\sigma_{r\theta}|_{\theta=\alpha} = 0$

When $\theta = 0$

$\sigma_\theta |_{\theta=0} = -q$

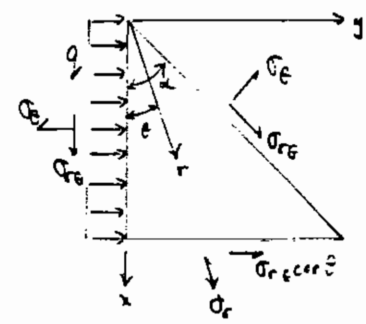
$\sigma_{r\theta}|_{\theta=0} = 0$

When $r = r$ $\quad \Sigma M_o = 0$

$-\frac{qr^2}{2} = \int_0^\alpha \sigma_{r\theta}\, r\cos\theta\, dy$

$-\frac{qr^2}{2} = \int_0^\alpha \sigma_{r\theta} \frac{r\cos\theta\, r d\theta}{\cos\theta}$

$-\frac{qr^2}{2} = \int_0^\alpha \sigma_{r\theta}\, r^2\, d\theta$



iii) $\sigma_r = \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = \frac{1}{r}[2r f(\theta)] + f''(\theta) = 2f(\theta) + f''(\theta)$

$\sigma_\theta = \frac{\partial^2\Phi}{\partial r^2} = \frac{\partial}{\partial r}[2r f(\theta)] = 2f(\theta)$

$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial\Phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\theta} = \frac{1}{r^2}[r^2 f'(\theta)] - \frac{1}{r}\frac{\partial}{\partial r}[r^2 f'(\theta)] = f'(\theta) - 2f'(\theta) = -f'(\theta)$

$\sigma_\theta|_{\theta=\alpha} = 2f(\alpha) = 2(A\cos 2\alpha + B\sin 2\alpha + C\alpha + D) = 0$

$\sigma_{r\theta}|_{\theta=\alpha} = -f'(\alpha) = 2A\sin 2\alpha - 2B\cos 2\alpha - C = 0$

$\sigma_\theta|_{\theta=0} = 2(A\cos 0 + B\sin 0 + D) = 2A + 2D = -q$

$\sigma_{r\theta}|_{\theta=0} = 2A\sin 0 - 2B\cos\theta - C = -2B - C = 0$

$$\begin{bmatrix} 2\cos 2\alpha & 2\sin 2\alpha & \alpha & 1 \\ 2\sin 2\alpha & -2\cos 2\alpha & -1 & 0 \\ 2 & 0 & 0 & 2 \\ 0 & -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -q \\ 0 \end{bmatrix}$$

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i) $\int_0^{\frac{\pi}{2}} (\sigma_r \sin\theta) r\, d\theta = \int_0^{\frac{\pi}{2}} \frac{2P\cos\theta}{\cancel{r}\pi} \sin\theta\, \cancel{r}\, d\theta$

$= \frac{P}{\pi} \int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta$

$= \frac{P}{\pi} \; -\frac{1}{2} \cos 2\theta \Big|_0^{\frac{\pi}{2}}$

$= -\frac{P}{2\pi} \left[(\cos\pi) - (\cos(0))\right]$

$= -\frac{P}{2\pi} (-1-1)$

$\int_0^{\frac{\pi}{2}} (\sigma_r \sin\theta) r\, d\theta = \frac{P}{\pi}$

ii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sigma_r \cos\theta) r\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2P\cos\theta}{\cancel{r}\pi} \cos\theta\, \cancel{r}\, d\theta$

$= \frac{2P}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta$

$= \frac{2P}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\, d\theta$

$= \frac{P}{\pi} \left[\theta + \frac{1}{2}\sin 2\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right]$

$= \frac{P}{\pi} \left(\frac{\pi}{2} + \frac{\pi}{2} \right)$

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sigma_r \cos\theta) r\, d\theta = P$

i $\checkmark$ (1)

$\frac{g \sin\alpha}{1(\sin\alpha - \alpha\cos\alpha)} - g$

$\Phi = f(\theta) r^2$

Show $\Phi = r^2 (A\cos 2\theta - B\sin 2\theta + C\theta + D)$

$\nabla^2 \nabla^2 \Phi = 0$

$\nabla^2 \Phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) (f(\theta) r^2)$

$= \frac{\partial}{\partial r}(2f(\theta) r) + \frac{1}{r}(2f(\theta) r) + \frac{\partial}{\partial \theta}(f'(\theta))$

$= 4f(\theta) + f''(\theta)$

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$\nabla^2\nabla^2 \Phi = \left(\cancel{\frac{\partial^2}{\partial r^2}} + \cancel{\frac{1}{r}\frac{\partial}{\partial r}} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) (4f(\theta) + f''(\theta))$

$= \frac{1}{r^2} \left[4 f^{2}(\theta) + f^{4}(\theta) \right]$

$\frac{1}{r^2} \left[4 f^2(\theta) + f^4(\theta) \right] = 0$

$\int 4 f'(\theta) + f^3(\theta) = a_0$

$4 f(\theta) + f^2(\theta) = a_0\theta + a_1 = g(\theta)$

w solution = homogeneous + particular

$f_g(\theta) = f_h(\theta) + f_p(\theta)$

genesow : $f_g(\theta) = 0 \Rightarrow 4f(\theta) + f^2(\theta) = 0$

$\hookrightarrow f_h(\theta) = A\cos 2\theta + B\sin 2\theta$

Chin Ci Esther Chau
1544-0499

Prob 7-7 $\sigma_\theta = \frac{1}{2}\sigma_0\left[\left(1+\frac{a^2}{r^2}\right)-\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right]$

For $r=a$, $\sigma_r = 0$

$\sigma_\theta\big|_{r=a} = \frac{1}{2}\sigma_0\left[(1+1)-(1+3)\cos 2\theta\right]$

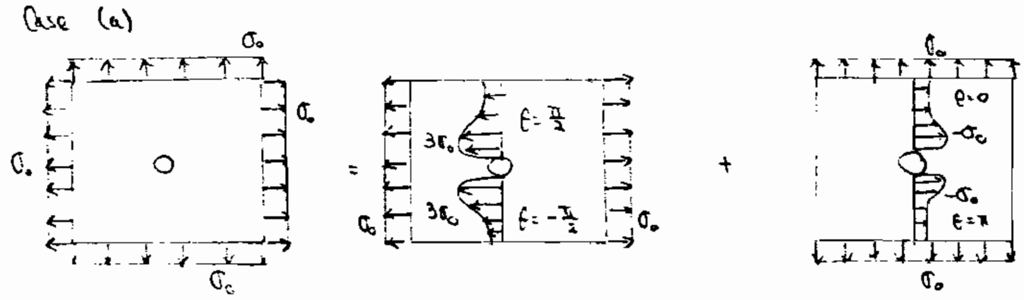
$= \frac{1}{2}\sigma_0\left(2-4\cos 2\theta\right)$

For $\theta = -\frac{\pi}{2}$, $\sigma_\theta\big|_{r=a} = 3\sigma_0$

For $\theta = 0$, $\sigma_\theta\big|_{r=a} = -\sigma_0$

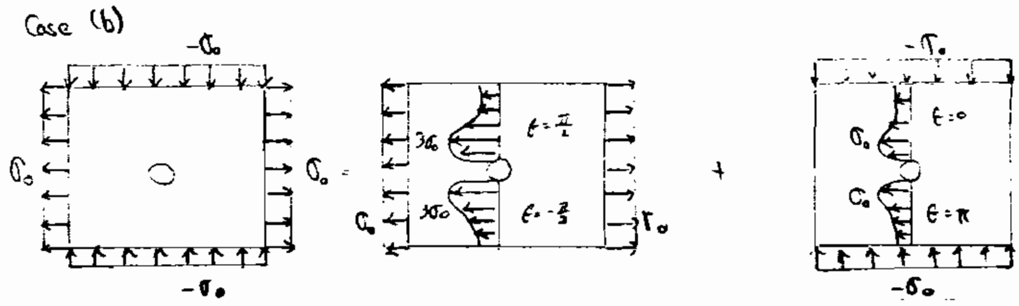
For $\theta = \frac{\pi}{2}$, $\sigma_\theta\big|_{r=a} = 3\sigma_0$

Case (a)



$\sigma_\theta = 3\sigma_0+(-\sigma_0) = 2\sigma_0$

Case (b)



$\sigma_\theta = 3\sigma_0 + (\sigma_0) = 4\sigma_0$

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