UNIVERSITY OF CALIFORNIA, BERKELEY College of Engineering

MICROMECHANICS (C236/C214) HW3

Problem 1: Find the Green's function for a both end clamped Euler-Bernoulli beam, i.e.

$$\frac{d^2}{dx^2} EI \frac{d^2 G(x,y)}{dx^2} = \delta(x-y), \quad \forall x, y \in (0, \ell) \tag{1}$$

and

$$G(0,y) = G(\ell, y) = 0, \quad G'(0,y) = G'(\ell,y) = 0 \ . \tag{2}$$

Problem 2: The Green's function, $G^\infty(\mathbf{x}, \mathbf{x}')$, satisfies the 2D Laplace equation,

$$\nabla^2 G^\infty(\mathbf{x}, \mathbf{x}') + \delta(\mathbf{x} - \mathbf{x}') = \mathbf{0}, \quad \forall \mathbf{x} \in \mathbb{R}^\mathbf{2} \tag{3}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} = \dfrac{\partial^2}{\partial x_\alpha \partial x_\alpha}$, $\alpha = 1, 2$. And $\delta(\mathbf{x} - \mathbf{x}') = \delta(x_1 - x_1')\delta(x_2 - x_2')$. Use Fourier transform method to derive

$$G^\infty(\mathbf{x} - \mathbf{x}') = -\frac{1}{2\pi} \ln|\mathbf{x} - \mathbf{x}'| \ . \tag{4}$$

Hints

$$\delta(\mathbf{x} - \mathbf{x}') = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\Big(i\boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')\Big) d\boldsymbol{\xi} \tag{5}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp(i(\xi_1 x_1 + \xi_2 x_2))}{\xi_1^2 + \xi_2^2} d\xi_1 \xi_2 = -\pi \ln(x_1^2 + x_2^2) \tag{6}$$

Problem 3:

For isotropic materials, elasticity tensor has the form

$$C_{ijk\ell} = \lambda \delta_{ij} \delta_{k\ell} + \mu(\delta_{i\ell}\delta_{jk} + \delta_{ik}\delta_{j\ell}) \tag{7}$$

Show

1.

$$K_{ik}(\boldsymbol{\xi}) = C_{ijk\ell}\xi_j \xi_\ell = (\lambda + \mu)\xi_i \xi_k + \mu \delta_{ik} \xi_j \xi_j \tag{8}$$

2. (Hint : use $e_{ijk} e_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$.)

$$\begin{aligned} N_{ij}(\boldsymbol{\xi}) &= \frac{1}{2} e_{ik\ell} e_{jmn} K_{km} K_{\ell n} \\ &= \mu \xi^2 ((\lambda + 2\mu)\delta_{ij}\xi^2 - (\lambda + \mu)\xi_i \xi_j) \end{aligned} \tag{9}$$

3.

$$D(\boldsymbol{\xi}) = \nu^2 (\lambda + 2\mu) \xi^6 \ . \tag{10}$$

Problem 4;

Consider one dimensional Helmhotz equation,

$$\frac{d^2u}{dx^2} + k^2u = \delta(|x-y|) \tag{11}$$

Hint: (The solution in the lecture notes may be wrong).

Problem 5:

In isotropic materials, the static Green's function of linear elasticity is

$$G_{ij}^{\infty}(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi\mu}\frac{\delta_{ij}}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{16\pi\mu(1-\nu)}\frac{\partial^2}{\partial x_i \partial x_j}|\mathbf{x}-\mathbf{x}'| \tag{12}$$

Let $\bar{\mathbf{x}} = \mathbf{x} - \mathbf{x}'$ and $\bar{x} = |\bar{\mathbf{x}}| = |\mathbf{x}-\mathbf{x}'|$. Show that for isotropic materials,

$$C_{j\ell mn}G_{ij,\ell} = \frac{-1}{8\pi(1-\nu)}\left\{(1-2\nu)\frac{\delta_{mi}\bar{x}_n + \delta_{ni}\bar{x}_m - \delta_{mn}\bar{x}_i}{\bar{x}^3} + 3\frac{\bar{x}_m\bar{x}_n\bar{x}_i}{\bar{x}^5}\right\} \tag{13}$$

where ν is the Poisson ratio, and μ, λ are the Lamé constants with

$$\lambda = \frac{2\mu\nu}{1-2\nu}, \quad \mu = \frac{\lambda(1-2\nu)}{2\nu}, \quad \nu = \frac{\lambda}{2(\lambda+\mu)} \tag{14}$$

Hint: ($C_{j\ell mn} = \lambda\delta_{j\ell}\delta_{mn} + \mu(\delta_{jm}\delta_{\ell n} + \delta_{jn}\delta_{\ell m})$).

Problem 6

Show

$$\int_{S^2} \delta''(n_k x_k) n_i n_i dS = 2\pi\nabla^2\left(\frac{1}{|\mathbf{x}|}\right) = -8\pi^2\delta(\mathbf{x}) \tag{15}$$

where n_k are components of the out-normal of S^2, and $\mathbf{x}$ is any position vector.

Hint: first show

$$\int_{S^2} \delta(n_k x_k) dS = \frac{2\pi}{|\mathbf{x}|} \tag{16}$$