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HW 10

Good!

20/20

1) $\beta_{ij} = \frac{1}{8\pi} \oint_C \left[-\varepsilon_{jkl} b_i R_{,ppl} + \varepsilon_{ikl} b_l R_{,ppj} + \frac{1}{1-\nu} \varepsilon_{mnk} b_n R_{,ijm} \right] dx'_k$

$e_{ij} = \frac{1}{2} (\beta_{ij} + \beta_{ji})$

$$= \frac{1}{16\pi} \oint_C \left[-(\varepsilon_{jkl} b_i + \varepsilon_{ikl} b_j) R_{,ppl} + (\varepsilon_{ikl} R_{,ppj} + \varepsilon_{jkl} R_{,ppi}) b_l + \frac{2}{1-\nu} b_n \varepsilon_{mnk} R_{,ijm} \right] dx'_k$$

From notes:

$$\varepsilon_{jkl} (b_i R_{,l} - b_l R_{,i}) = (\varepsilon_{kst} \delta_{ji} - \varepsilon_{jst} \delta_{ki}) b_s R_{,t}$$

$$\varepsilon_{ikl} (b_j R_{,l} - b_l R_{,j}) = (\varepsilon_{kst} \delta_{ij} - \varepsilon_{ist} \delta_{kj}) b_s R_{,t}$$

Subst. above

$$e_{ij} = \frac{1}{8\pi} \oint \left[-b_s R_{,ppt} \left\{ \varepsilon_{kst} \delta_{ij} - \frac{1}{2} \varepsilon_{ist} \delta_{kj} - \frac{1}{2} \varepsilon_{jst} \delta_{ki} \right\} + \frac{1}{1-\nu} \varepsilon_{mnk} b_n R_{,ijm} \right] dx'_k$$

$\sigma_{ij} = C_{ijkl} e_{kl}$ & $C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$

$$e_{kl} = \frac{1}{8\pi} \oint \left[-b_s R_{,ppt} \left\{ \varepsilon_{ost} \delta_{kl} - \frac{1}{2} \varepsilon_{kst} \delta_{ol} - \frac{1}{2} \varepsilon_{lst} \delta_{ok} \right\} + \frac{1}{1-\nu} \varepsilon_{mno} b_n R_{,klm} \right] dx'_o$$

$\lambda \delta_{ij} \delta_{kl} e_{kl}$

$$= \frac{\lambda}{8\pi} \oint \left[-b_s R_{,ppt} \left\{ 3 \underbrace{\varepsilon_{ost}}_{\delta_{ij}} - \frac{1}{2} \overbrace{\varepsilon_{kst} \delta_{ok}}^{\varepsilon_{ost}} \delta_{ij} - \frac{1}{2} \overbrace{\varepsilon_{kst} \delta_{ok} \delta_{ij}}^{\varepsilon_{ost}} \right\} + \frac{1}{1-\nu} \varepsilon_{mno} b_n R_{,kkm} \delta_{ij} \right] dx'_o$$

$$= \frac{\lambda}{8\pi} \oint \left[-2 b_s R_{,ppt} \delta_{ij} \varepsilon_{ost} + \frac{b_n}{1-\nu} \varepsilon_{mno} R_{,kkm} \delta_{ij} \right] dx'_o$$

$\mu \, \delta_{ik} \delta_{jl} e_{kl}$

$$= \frac{\mu}{8\pi} \oint \Big[-b_s R_{,ppt} \Big\{ \epsilon_{ost} \delta_{ij} - \frac{1}{2} \epsilon_{ist} \delta_{oj} - \frac{1}{2} \epsilon_{jst} \delta_{oi} \Big\} + \frac{b_n}{1-\nu} \epsilon_{mno} R_{,ijm} \Big] dx_o'$$

$\mu \, \delta_{il} \delta_{jk} e_{kl}$

$$= \frac{\mu}{8\pi} \oint \Big[-b_s R_{,ppt} \Big\{ \epsilon_{ost} \delta_{ij} - \frac{1}{2} \epsilon_{jst} \delta_{oi} - \frac{1}{2} \epsilon_{ist} \delta_{oj} \Big\} + \frac{b_n}{1-\nu} \epsilon_{mno} R_{,ijm} \Big] dx_o'$$

$\therefore \; C_{ijkl} e_{kl}$

$$= \frac{1}{8\pi} \oint \Big[-2(\lambda+\mu) b_s R_{,ppt} \delta_{ij} \epsilon_{ost} + \mu b_s R_{,ppt} \left(\epsilon_{ist} \delta_{oj} + \epsilon_{jst} \delta_{oi} \right) + \frac{\lambda b_n}{1-\nu} \epsilon_{mno} \delta_{ij} R_{,kkm} + \frac{2\mu b_n}{1-\nu} \epsilon_{mno} R_{,ijm} \Big] dx_o'$$

$s \longrightarrow n \qquad t \longrightarrow m$

$$= \frac{1}{8\pi} \oint \Big[-2(\lambda+\mu) b_n R_{,ppm} \delta_{ij} \epsilon_{onm} + \mu b_n R_{,ppm} \left(\epsilon_{inm} \delta_{oj} + \epsilon_{jnm} \delta_{oi} \right) + \frac{\lambda b_n}{1-\nu} \epsilon_{mno} \delta_{ij} R_{,ppm} + \frac{2\mu b_n}{1-\nu} \epsilon_{mno} R_{,ijm} \Big] dx_o'$$

$$= \frac{1}{8\pi} \oint \Big[\mu b_n R_{,ppm} \left(\epsilon_{inm} dx_j' + \epsilon_{jnm} dx_i' \right) + \frac{2\mu b_n}{1-\nu} \epsilon_{mno} R_{,ijm} dx_o' + b_n \epsilon_{mno} \delta_{ij} R_{,ppm} \left(\frac{\lambda}{1-\nu} + 2(\lambda+\mu) \right) dx_o' \Big]$$

$$= \frac{\mu}{4\pi} \oint \Big[\frac{b_n}{2} R_{,ppm} \left(\epsilon_{jnm} dx_i' + \epsilon_{inm} dx_j' \right) + \frac{b_n}{1-\nu} \epsilon_{mno} R_{,ijm} dx_o' + \frac{b_n}{2} \epsilon_{mno} \delta_{ij} R_{,ppm} \left(\frac{\lambda}{\mu} \cdot \frac{1}{1-\nu} + 2\frac{\lambda}{\mu} + 2 \right) dx_o' \Big]$$

$$\frac{\lambda}{\mu} = \frac{2\nu}{1-2\nu}.$$

$$\Rightarrow \frac{\lambda}{\mu}\frac{1}{1-\nu} + 2\frac{\lambda}{\mu} + 1 = \frac{2\nu}{(1-\nu)(1-2\nu)} + \frac{4\nu}{(1-2\nu)} + 2$$

$$= \frac{2\nu + 4\nu(1-\nu) + 2(1-\nu)(1-2\nu)}{(1-\nu)(1-2\nu)}.$$

$$= \frac{\cancel{2\nu} + \cancel{4\nu} - \cancel{4\nu^2} + 2 + \cancel{4\nu^2} \cancel{-6\nu}}{(1-\nu)(1-2\nu)} = \frac{2}{(1-\nu)(1-2\nu)}$$

$$\therefore \sigma_{ij} = \frac{\mu}{4\pi}\oint\left[\frac{b_n}{2} R_{,ppm}\left(\varepsilon_{jnm}\, dx_i' + \varepsilon_{inm}\, dx_j'\right) + \frac{b_n}{(1-\nu)}\varepsilon_{mno}\left\{R_{,ijm} + \frac{1}{(1-2\nu)}\delta_{ij} R_{,ppm}\right\} dx_o'\right]$$

2]

$$\sigma = \frac{\mu}{4\pi}\oint_C (b \times \nabla)\frac{1}{R} \otimes dl + \frac{\mu}{4\pi}\oint_C dl \otimes (b \times \nabla)\frac{1}{R}$$

$$- \frac{\mu}{4\pi(1-\nu)}\oint_C \nabla\cdot(b \times dl)\cdot(\nabla \otimes \nabla - 1\nabla^2)R \qquad [\text{Eq. } 7.193]$$

$$\delta F = \left(\delta\sigma\Big|_{\substack{b_2\\ dl_2}} \cdot b_1\right) \times dl_1 \qquad ; \quad \frac{1}{R} = \frac{1}{2}\nabla^2 R$$

Evaluating term by term:

a) $$\left[\left\{(b_2 \times \nabla)\frac{1}{R} \otimes dl_2\right\}\cdot b_1\right] \times dl_1$$

$$= \frac{1}{2}\left[\left\{(b_2 \times \nabla(\nabla^2 R) \otimes dl_2\right\}\cdot b_1\right] \times dl_1$$

$$= \frac{1}{2}(dl_2 \cdot b_1)\left\{b_2 \times \nabla(\nabla^2 R)\right\} \times dl_1$$

$$= \frac{1}{2}\left\{\left[b_2 \times \nabla(\nabla^2 R)\right] \times dl_1\right\}(b_1 \cdot dl_2)$$

b) $$\frac{1}{2}\left[\left\{dl_2 \otimes (b_2 \times \nabla(\nabla^2 R))\right\}\cdot b_1\right] \times dl_1$$

$$= \frac{1}{2}\left[\left\{b_2 \times \nabla(\nabla^2 R)\right\}\cdot b_1\right](dl_2 \times dl_1)$$

$$= \frac{1}{2}\left[(b_1 \times b_2)\cdot\nabla(\nabla^2 R)\right](dl_2 \times dl_1)$$

$$= \frac{1}{2}\left[(b_2 \times b_1)\cdot\nabla(\nabla^2 R)\right](dl_1 \times dl_2)$$

c) $$\left[\left\{\nabla\cdot(b_2 \times dl_2)T\right\}\cdot b_1\right] \times dl_1 \qquad T = (\nabla \otimes \nabla)R$$

$$= \nabla\cdot(b_2 \times dl_2)(T b_1 \times dl_1)$$

$$= -\nabla\cdot(b_2 \times dl_2)(dl_1 \times T b_1)$$

d) $\left[\left\{\nabla\cdot(\underline{b}_2\times d\underline{l}_2)\ \nabla^2 R\right\}\cdot\underline{b}_1\right]\times d\underline{l}_1$

$$= \left[\nabla(\nabla^2 R)\cdot(\underline{b}_2\times d\underline{l}_2)\cdot\underline{b}_1\right)\times d\underline{l}_1$$

$$= \left[(\underline{b}_2\times d\underline{l}_2)\cdot\nabla(\nabla^2 R)\right](\underline{b}_1\times d\underline{l}_1)$$

$$= -\left[(\underline{b}_2\times d\underline{l}_2)\cdot\nabla(\nabla^2 R)\right](d\underline{l}_1\times\underline{b}_1)$$

Combining all the terms together.

$$\delta\underline{F} = \frac{\mu}{8\pi}\int_{y_1}^{y_2}\left[(\underline{b}_2\times\underline{b}_1)\cdot\nabla(\nabla^2 R)\right](d\underline{l}_1\times d\underline{l}_2) + \frac{\mu}{8\pi}\int_{y_1}^{y_2}\left\{\left[\underline{b}_2\times\nabla(\nabla^2 R)\right]\times d\underline{l}_1\right\}(\underline{b}_1\cdot d\underline{l}_2)$$

$$+\frac{\mu}{4\pi(1-\nu)}\int_{y_1}^{y_2}\nabla\cdot(\underline{b}_2\times d\underline{l}_2)(d\underline{l}\times\underline{I}\,\underline{b}_1) - \frac{\mu}{4\pi(1-\nu)}\int_{y_1}^{y_2}\left[(\underline{b}_2\times d\underline{l}_2)\cdot\nabla(\nabla^2 R)\right](d\underline{l}_1\times\underline{b}_1)$$