PROBLEM 1

Show $\underset{\sim}{1}^{(4)} : \underset{\sim}{A} = \underset{\sim}{A}$

$\underset{\sim}{1}^{(4s)} : \underset{\sim}{A} = \frac{1}{2}(\underset{\sim}{A} + \underset{\sim}{A}^T)$

$\underset{\sim}{1}^{(4a)} : \underset{\sim}{A} = \frac{1}{2}(\underset{\sim}{A} - \underset{\sim}{A}^T)$

- $\underset{\sim}{1}^{(4)} : \underset{\sim}{A} = \left[\delta_{ik}\delta_{jl}(\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l)\right] : \left[a_{rs}(\underline{e}_r \otimes \underline{e}_s)\right]$

$= \delta_{ik}\delta_{jl}\, a_{rs}\, \delta_{kr}\delta_{ls}\, (\underline{e}_i \otimes \underline{e}_j)$

$= \delta_{ik}\delta_{jl}\, a_{kl}\, (\underline{e}_i \otimes \underline{e}_j)$

$= a_{ij}\, (\underline{e}_i \otimes \underline{e}_j)$

$= \underset{\sim}{A}$

- $\underset{\sim}{1}^{(4s)} : \underset{\sim}{A} = \left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})(\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l)\right] : \left[a_{rs}(\underline{e}_r \otimes \underline{e}_s)\right]$

$= \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\, a_{rs}\, \delta_{kr}\delta_{ls}\, (\underline{e}_i \otimes \underline{e}_j)$

$= \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\, a_{kl}\, (\underline{e}_i \otimes \underline{e}_j)$

$= \frac{1}{2}(a_{ij} + a_{ji})(\underline{e}_i \otimes \underline{e}_j)$

$= \frac{1}{2}\left(a_{ij}(\underline{e}_i \otimes \underline{e}_j) + a_{ji}(\underline{e}_i \otimes \underline{e}_j)\right)$

$= \frac{1}{2}(\underset{\sim}{A} + \underset{\sim}{A}^T)$

- $\underset{\sim}{1}^{(4a)} : \underset{\sim}{A} = \left[\frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})(\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l)\right] : \left[a_{rs}(\underline{e}_r \otimes \underline{e}_s)\right]$

$= \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\, a_{rs}\, \delta_{kr}\delta_{ls}\, (\underline{e}_i \otimes \underline{e}_j)$

$= \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\, a_{kl}\, (\underline{e}_i \otimes \underline{e}_j)$

$= \frac{1}{2}(a_{ij} - a_{ji})(\underline{e}_i \otimes \underline{e}_j)$

$= \frac{1}{2}\left(a_{ij}(\underline{e}_i \otimes \underline{e}_j) - a_{ji}(\underline{e}_i \otimes \underline{e}_j)\right)$

$= \frac{1}{2}(\underset{\sim}{A} - \underset{\sim}{A}^T)$

PROBLEM 2

Show $\quad e_{ijk}\, e_{ijk} = 3! = 6$

$e_{ijk}\, e_{ijl} = 2\,\delta_{kl}$

$e_{ijk}\, e_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}$

- $e_{ijk}\, e_{ijk} = \begin{vmatrix} \delta_{ii} & \delta_{ij} & \delta_{ik} \\ \delta_{ji} & \delta_{jj} & \delta_{jk} \\ \delta_{ki} & \delta_{kj} & \delta_{kk} \end{vmatrix}$

$= \delta_{ii}\left(\delta_{jj}\,\delta_{kk} - \delta_{kj}\,\delta_{jk}\right) - \delta_{ij}\left(\delta_{ji}\,\delta_{kk} - \delta_{ki}\,\delta_{jk}\right) + \delta_{ik}\left(\delta_{ji}\,\delta_{kj} - \delta_{ki}\,\delta_{jj}\right)$

$= \delta_{ii}\,\delta_{jj}\,\delta_{kk} - \delta_{ii}\,\delta_{jj} - \delta_{ii}\,\delta_{kk} + \delta_{ii} + \delta_{ii} - \delta_{ii}\,\delta_{jj}$

$= 3 \cdot 3 \cdot 3 - 3 \cdot 3 - 3 \cdot 3 + 3 + 3 - 3 \cdot 3$

$= 6$

- $e_{ijk}\, e_{ijl} = \begin{vmatrix} \delta_{ii} & \delta_{ij} & \delta_{il} \\ \delta_{ji} & \delta_{jj} & \delta_{jl} \\ \delta_{ki} & \delta_{kj} & \delta_{kl} \end{vmatrix}$

$= \delta_{ii}\left(\delta_{jj}\,\delta_{kl} - \delta_{kj}\,\delta_{jl}\right) - \delta_{ij}\left(\delta_{ji}\,\delta_{kl} - \delta_{ki}\,\delta_{jl}\right) + \delta_{il}\left(\delta_{ji}\,\delta_{kj} - \delta_{ki}\,\delta_{jj}\right)$

$= \delta_{kl}\left(\delta_{ii}\,\delta_{jj} - \delta_{ii} - \delta_{ii} + 1 + 1 - \delta_{jj}\right)$

$= \delta_{kl}\,(3 \cdot 3 - 3 - 3 + 2 - 3$

$= 2 \cdot \delta_{kl}$

- $e_{ijk}\, e_{ilm} = \begin{vmatrix} \delta_{ii} & \delta_{il} & \delta_{im} \\ \delta_{ji} & \delta_{jl} & \delta_{jm} \\ \delta_{ki} & \delta_{kl} & \delta_{km} \end{vmatrix}$

$= \delta_{ii}\left(\delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}\right) - \delta_{il}\left(\delta_{ji}\,\delta_{km} - \delta_{jm}\,\delta_{ki}\right) + \delta_{im}\left(\delta_{ji}\,\delta_{kl} - \delta_{jl}\,\delta_{ki}\right)$

$= \left(\delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}\right) \cdot \left(\delta_{ii} - 1 + (-1)\right)$

$= \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}$

PROBLEM 3

Show that a) $\mathbb{E}^{(1)} + \mathbb{E}^{(2)} = \mathbf{1}^{(4s)}$

b) $\mathbb{E}^{(1)} : \mathbb{E}^{(1)} = \mathbb{E}^{(1)}$

c) $\mathbb{E}^{(2)} : \mathbb{E}^{(2)} = \mathbb{E}^{(2)}$

d) $\mathbb{E}^{(2)} : \mathbb{E}^{(1)} = = \mathbb{E}^{(1)} : \mathbb{E}^{(2)} = \mathbb{O}$

a) $$\mathbb{E}^{(1)} + \mathbb{E}^{(2)} = \left[\tfrac{1}{3}\ \cancel{\mathbf{1}^{(2)} \otimes \mathbf{1}^{(2)}}\right] + \left[\mathbf{1}^{(4s)} - \cancel{\tfrac{1}{3}\ \mathbf{1}^{(2)} \otimes \mathbf{1}^{(2)}}\right] = \mathbf{1}^{(4s)}$$

b) $$\mathbb{E}^{(1)} : \mathbb{E}^{(1)} = \left[\tfrac{1}{3}\ \mathbf{1}^{(2)} \otimes \mathbf{1}^{(2)}\right] : \left[\tfrac{1}{3}\ \mathbf{1}^{(2)} \otimes \mathbf{1}^{(2)}\right]$$
$$= \left[\tfrac{1}{3}\ \delta_{ij}\delta_{rs} (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_r \otimes \mathbf{e}_s)\right] : \left[\tfrac{1}{3}\ \delta_{kl}\delta_{tu} (\mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_t \otimes \mathbf{e}_u)\right]$$
$$= \tfrac{1}{9}\ \underbrace{\delta_{ij}\delta_{rs}\delta_{kl}\delta_{tu}\delta_{rt}\delta_{su}}\ (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)$$
$$= \delta_{ij}\delta_{kl}\delta_{ts}\delta_{ts} = \delta_{ij}\delta_{kl}\delta_{ss} = 3 \cdot \delta_{ij}\delta_{kl}$$
$$= \tfrac{1}{3}\ \delta_{ij}\delta_{kl}\ (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) = \mathbb{E}^{(1)}$$

c) $$\mathbb{E}^{(2)} : \mathbb{E}^{(2)} = \left[\mathbf{1}^{(4s)} - \mathbb{E}^{(1)}\right] : \left[\mathbf{1}^{(4s)} - \mathbb{E}^{(1)}\right]$$
$$= \mathbf{1}^{(4s)} : \mathbf{1}^{(4s)} - \mathbb{E}^{(1)} : \mathbf{1}^{(4s)} - \mathbf{1}^{(4s)} : \mathbb{E}^{(1)} + \mathbb{E}^{(1)} : \mathbb{E}^{(1)}$$
$$= \mathbf{1}^{(4s)} : \mathbf{1}^{(4s)} - 2\, \mathbb{E}^{(1)} : \mathbf{1}^{(4s)} + \mathbb{E}^{(1)} : \mathbb{E}^{(1)}$$

where • $\mathbb{E}^{(1)} : \mathbb{E}^{(1)} = \mathbb{E}^{(1)}$ by (1)

• $$\mathbb{E}^{(1)} : \mathbf{1}^{(4s)} = \left[\tfrac{1}{3}\ \delta_{ij}\delta_{rs} (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_r \otimes \mathbf{e}_s)\right] : \left[\tfrac{1}{2}(\delta_{kt}\delta_{lu} + \delta_{ku}\delta_{lt})(\mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_t \otimes \mathbf{e}_u)\right]$$
$$= \tfrac{1}{6}\ \delta_{ij}\delta_{rs}(\delta_{kt}\delta_{lu} + \delta_{ku}\delta_{lt})\,\delta_{rt}\delta_{su}\ (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)$$
$$= \tfrac{1}{6}\ \delta_{ij}\delta_{rs}(\delta_{kr}\delta_{ls} + \delta_{ks}\delta_{lr})\ (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)$$
$$= \tfrac{1}{6}\ (\delta_{ij}\delta_{kl} + \delta_{ij}\delta_{kl})\ (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)$$
$$= \tfrac{1}{3}\ \delta_{ij}\delta_{kl}\ (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) = \mathbb{E}^{(1)}$$

$$\bullet\ \underset{\sim}{1}^{(4s)} : \underset{\sim}{1}^{(4s)} = \left[\tfrac{1}{2}\left(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}\right)\left(\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_r \otimes \underline{e}_s\right)\right] : \left[\tfrac{1}{2}\left(\delta_{kt}\delta_{lu} + \delta_{ku}\delta_{lt}\right)\left(\underline{e}_k \otimes \underline{e}_l \otimes \underline{e}_t \otimes \right.\right.$$

$$= \tfrac{1}{4}\left(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}\right)\left(\delta_{kt}\delta_{lu} + \delta_{ku}\delta_{lt}\right)\delta_{rt}\delta_{su}\ \left(\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l\right)$$

$$= \tfrac{1}{4}\left(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}\right)\left(\delta_{kr}\delta_{ls} + \delta_{ks}\delta_{lr}\right)\ \left(\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l\right)$$

$$= \tfrac{1}{4}\left(\delta_{ir}\delta_{js}\delta_{kr}\delta_{ls} + \delta_{is}\delta_{jr}\delta_{kr}\delta_{ls} + \delta_{ir}\delta_{js}\delta_{ks}\delta_{lr} + \delta_{is}\delta_{jr}\delta_{ks}\delta_{lr}\right)\left(\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}\right.$$

$$= \tfrac{1}{4}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\right)\ \left(\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l\right)$$

$$= \tfrac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\ \left(\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l\right) = \underset{\sim}{1}^{(4s)} \qquad (5$$

Inserting (3), (4) and (5) into (2) results in

$$\underset{\sim}{E}^{(2)} : \underset{\sim}{E}^{(2)} = \underset{\sim}{1}^{(4s)} - 2 \cdot \underset{\sim}{E}^{(1)} + \underset{\sim}{E}^{(1)}$$

$$= \underset{\sim}{1}^{(4s)} - \underset{\sim}{E}^{(1)}$$

$$= \underset{\sim}{E}^{(2)}$$

d) $$\underset{\sim}{E}^{(2)} : \underset{\sim}{E}^{(1)} = \underset{\sim}{E}^{(1)} : \underset{\sim}{E}^{(2)}$$

$$= \underset{\sim}{E}^{(1)} : \left(\underset{\sim}{1}^{(4s)} - \underset{\sim}{E}^{(1)}\right)$$

$$= \underbrace{\underset{\sim}{E}^{(1)} : \underset{\sim}{1}^{(4s)}}_{= \underset{\sim}{E}^{(1)} \text{ by } (4)} - \underbrace{\underset{\sim}{E}^{(1)} : \underset{\sim}{E}^{(1)}}_{= \underset{\sim}{E}^{(1)} \text{ by } (1)}$$

$$= \underset{\sim}{E}^{(1)} - \underset{\sim}{E}^{(1)} = \underset{\sim}{0}$$