Problem 1 (40 points)
Consider a square $3 \times 3$ real number matrix $A$, whose eigenvalue problem is given as

$$ A = \lambda I, $$

(1)

where $I$ is the square unit matrix.

One can find its eigenvalues by solving the following characteristic polynomial equation,

$$ \det[A - \lambda I] = -\lambda^3 + I_1(A)\lambda^2 - I_2(A)\lambda + I_3(A) = 0, $$

(1) (2)

where

$$ I_1(A) = A_{11} + A_{22} + A_{33}, \quad I_2(A) = \frac{1}{2}(I_1^2(A) - I_3(A^2)), \quad \text{and} \quad I_3 = \det(A). $$

Show that the following matrix equation holds

$$ -A^3 + I_1(A)A^2 - I_2(A)A + I_3(A)I = 0, $$

(2)

which is the so-called Cayley-Hamilton theorem.

Hint:
Multiply the equation (1) with the second order identity matrix $I$.

Problem 2 (60 points)
Define an integration as

$$ I(y(x)) = \int_0^\ell \sqrt{1 + (y'(x))^2} \, dx, \quad \text{where} \quad y' = \frac{dy}{dx} $$

where $y(x)$ is a smooth real function defined in $[0, \ell]$. Obviously, the value of integration depends on the selection of the function $y(x)$ (In fact, it is a functional, but you do not need that knowledge to solve the problem).

Let

$$ I(y(x) + \epsilon w(x)) = \int_0^\ell \sqrt{1 + (y'(x) + \epsilon w'(x))^2} \, dx, $$

where $\epsilon > 0$ is a real number, and $w(x)$ is another real function.
(1) For given functions \(y(x)\) and \(w(x)\), calculate the following quantity by taking the limit on \(\epsilon\),

\[
\delta I := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( I(y(x) + \epsilon w(x)) - I(y(x)) \right)
\]

(2) Define a real function: \(f(\epsilon) := I(y(x) + \epsilon w(x))\). Expand it into the Taylor series of \(\epsilon\) at \(\epsilon = 0\) for the first three terms, i.e.

\[
f(\epsilon) = f(0) + f'(0)\epsilon + \frac{1}{2} f''(0)\epsilon^2 + o(\epsilon^2) .
\]
Problem 1:
Consider the following linear system of ordinary differential equations,
\[
\frac{dy_1}{dt} = y_2(t) + y_3(t) \quad (1)
\]
\[
\frac{dy_2}{dt} = y_1(t) + y_3(t) \quad (2)
\]
\[
\frac{dy_3}{dt} = y_1(t) + y_2(t) . \quad (3)
\]
Find the general solution for the following ordinary differential equations. (50 points)

Problem 2:
Show that the Laplace equation in two-dimensional space,
\[
\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0,
\]
(4)
can be converted into a form of polar coordinates as
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} = 0 .
\]
where
\[
x = r \cos \theta, \quad y = r \sin \theta; \quad \text{or} \quad r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}.
\]
If the problem is axi-symmetric, i.e. \( \Phi = \Phi(r) \), what is the basic form of the solution for the equation. That is the solution satisfying
\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{d\Phi}{dr} \right) = 0 .
\]
(50 points)
PH.D. PRELIMINARY EXAMINATION

MATHEMATICS

Problem 1 (50 points)
Consider the following $3 \times 3$ matrix,
\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

(1) Find all the eigenvalues of the matrix;
(2) Find the corresponding eigenvectors.

Problem 2 (50 points)
Consider the following equilibrium equation of an elastic bar,
\[
\frac{d}{dx} \left( E(y) A \frac{du^\varepsilon}{dx} \right) + b(x) = 0, \quad 0 < x < L
\]

with boundary conditions,
\[
\frac{du^\varepsilon(0)}{dx} = 0, \quad \text{and} \quad P(L) = E(y) A \frac{du^\varepsilon}{dx} \bigg|_{x=L} = \bar{P}.
\]

Let
\[
\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{1}{\varepsilon} \frac{\partial}{\partial y}
\]
where $\varepsilon << 1$ is a small length scale parameter. Assume that
\[
\frac{d}{dx}(x) = u_0(x, y) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + \cdots
\]

Substitute Eqs. (3) and (4) into Eq. (1) and the boundary condition (2), and expand the equilibrium equation of the elastic bar into a series expansion of $\varepsilon$, i.e. group every terms that have same exponent of $\varepsilon$ together. Find the first three equations that corresponding to the lowest three exponents (scales) of $\varepsilon$, i.e. $\varepsilon^{-2}$, $\varepsilon^{-1}$, and $\varepsilon^0$. 
Problem 1 (50 points)
Consider the following linear equation,
\[
\begin{bmatrix}
(x - 1) & 0 & x \\
0 & 2 & 0 \\
x & 0 & (x - 1)
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}
= \begin{bmatrix}
x^2 \\
5 \\
x^2 - x + 1/4
\end{bmatrix}
\]
(1) Find all the eigenvalues of the coefficient matrix;
(2) Find all corresponding eigenvectors;
(3) Discuss the solution of the linear algebraic equation \( \{y_i\} \) when \( x \to 1/2 \).

Problem 2 (50 points) Consider the following Euler-Bernoulli beam equation,
\[
\frac{d^2}{dx^2} EI(x) \frac{d^2 v}{dx^2} + kv(x) = 0, \quad 0 < x < L
\]
with the boundary conditions,
\[
v(0) = 0, \quad v'(0) = 0; \quad V(L) := \left. \frac{d}{dx} EI(x) \frac{d^2 v}{dx^2} \right|_{x=L} = -P_L, \quad M(L) := \left. EI(x) \frac{d^2 v}{dx^2} \right|_{x=L} = 0.
\]
Choose an arbitrary continuous function \( w(x) \in H^2([0, L]) \) and
\[
w(0) = 0, \quad \left. \frac{dw}{dx} \right|_{x=0} = 0.
\]
(1) Use the weak form of the differential equation to solve \( w(L) \) in terms of definite integrations of \( v(x) \) and \( w(x) \) and their second derivatives. Note that \( E, I(x) \) and \( k \) are either given constants or known function;
(2) Choose \( w(x) = v(x) \) and solve \( v(L) \) in terms of \( E, P_L, L \) and \( I_0 \) by assuming that \( I(x) = I_0(L-x) \) and \( k = 0 \).

Hint: First observe and verify that \( M(x) := EI(x) \frac{d^2 v}{dx^2} = P_L(L - x) \).
PH.D. PRELIMINARY EXAMINATION

MATHEMATICS

Problem 1: (50 points)
Consider the following heat conduction equation of an one-dimensional body in a region \((0 \leq X \leq L)\),

\[
\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial X^2}, \quad 0 < X < L;
\]

with boundary conditions,

\[
\theta(0, t) = 0, \quad \text{and} \quad \theta(L, t) = 0.
\]

1. Find the general expression of the solution;
2. If the initial condition is given as

\[
\theta(X, 0) = A \sin \frac{\pi x}{L}, \quad A \text{ is a given constant}
\]

Determine the exact solution of the problem.

Problem 2: (50 points)

Consider the following matrix,

\[
[A] = \begin{bmatrix}
3 & 0 & 2 \\
0 & -1 & 0 \\
2 & 0 & 0
\end{bmatrix}
\]

(1) Find all the eigenvalues;
(2) Find all the eigenvectors;
(3) Find a transformation matrix \([Q]\) such that

\[
[B] = [Q]^T [A] [Q]
\]

is a diagonal matrix.
Problem #1
Let \( u(x, y) \) denote the deflection of a rectangular membrane of sides \( a \) and \( b \) (i.e., \( 0 \leq x \leq a \) and \( 0 \leq y \leq b \) for a plane Cartesian system). This deflection satisfies the partial differential equation

\[
c \Delta u + q(x, y) = 0,
\]

where \( c \) is a given parameter (tension in the membrane), \( \Delta \) denotes the Laplacian in the \( xy \) system, and \( q(x, y) \) is the transversal loading distributed per unit area.

1. Find \( u(x, y) \) if the membrane is fixed along its sides, and the applied load is constant \( q(x, y) = q_0 \).

2. What is the maximum deflection of the membrane for the case in Item 1? Particularize your answer for a square membrane \( a = b \), indicating also an (educated) approximate value.

Problem #2
Find the matrix \( X \) satisfying the relation

\[
\begin{bmatrix}
55 & 45 \\
45 & 55
\end{bmatrix} = 10^X
\]

(i.e. "ten to the power \( X \)).