Linear Eulerian Model of En route air traffic flow

Alex Bayen
Department of Civil and Environmental Engineering
University of California at Berkeley

Modeling, optimization and software in Air Traffic Management Workshop

45th IEEE Conference on Decision and Control, December 12th, 2006, San Diego, CA

Outline

1. Introduction: Eulerian/Langrangian; Micro/Macro-aggregate

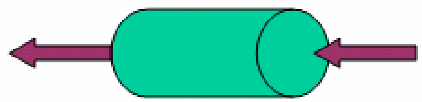
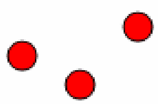
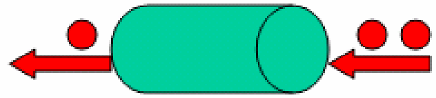
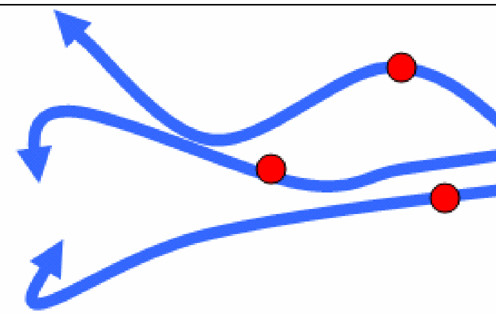
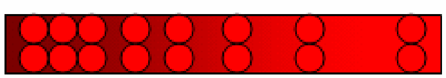
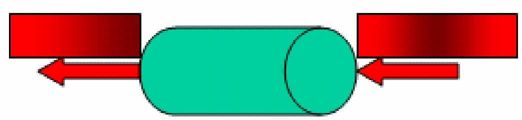
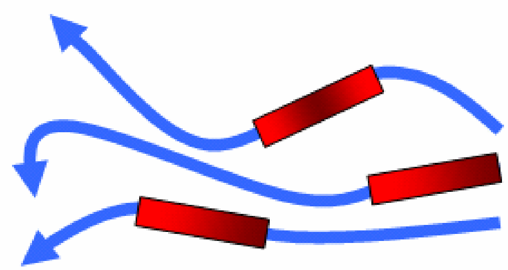
2. Delay model for TFM: multicommodity cell transmission
    1. Automated graph topology model building
    2. Aggregate travel time estimation

3. LTI models of the NAS
    1. Standard LTI formulation
    2. Constrained optimization formulations
    3. Practical implementation

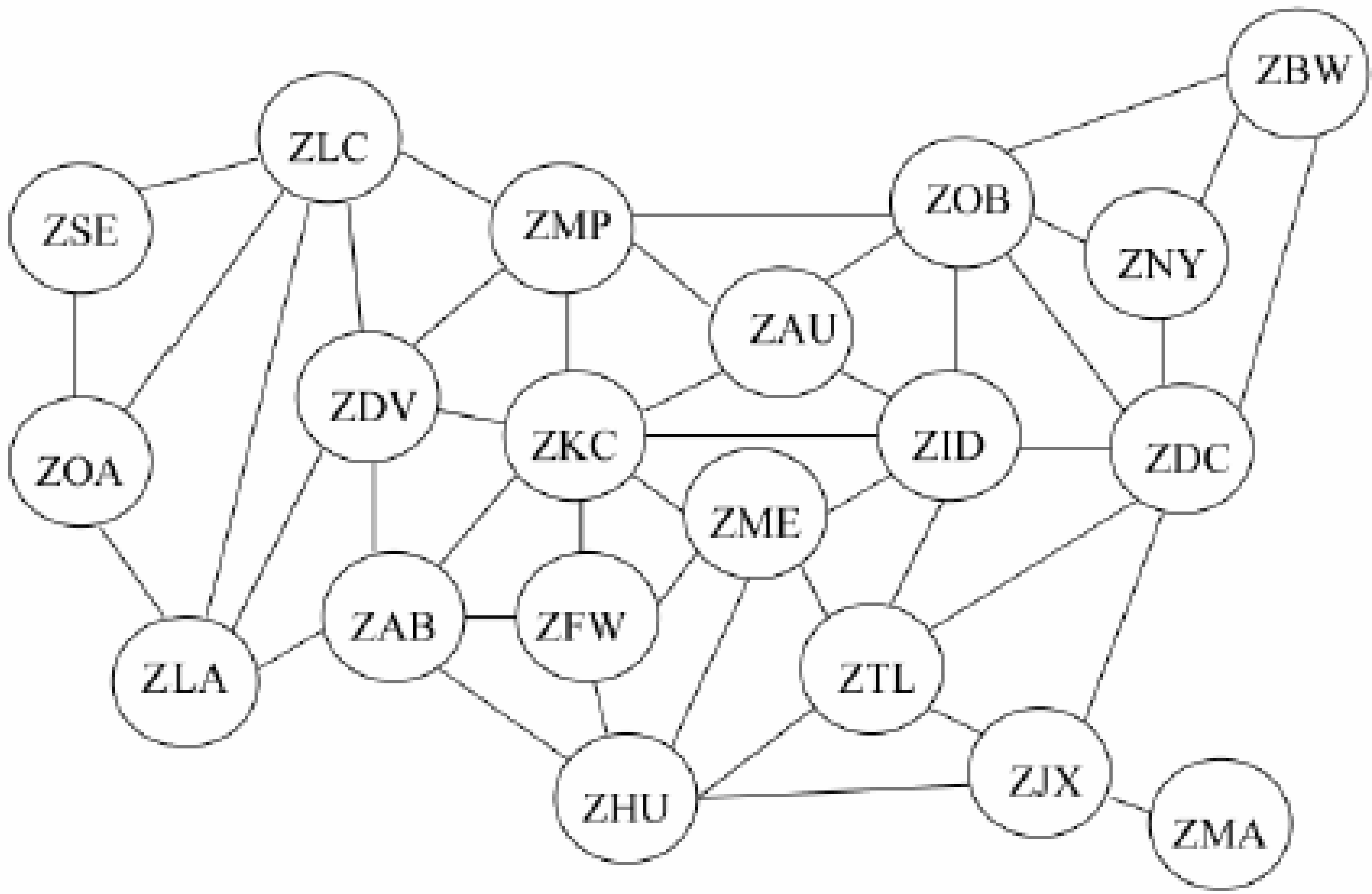
4. Beyond Mixed Integer Linear Programming (MILP)

5. Future directions

Eulerian / Lagrangian; micro / macro (aggregate)

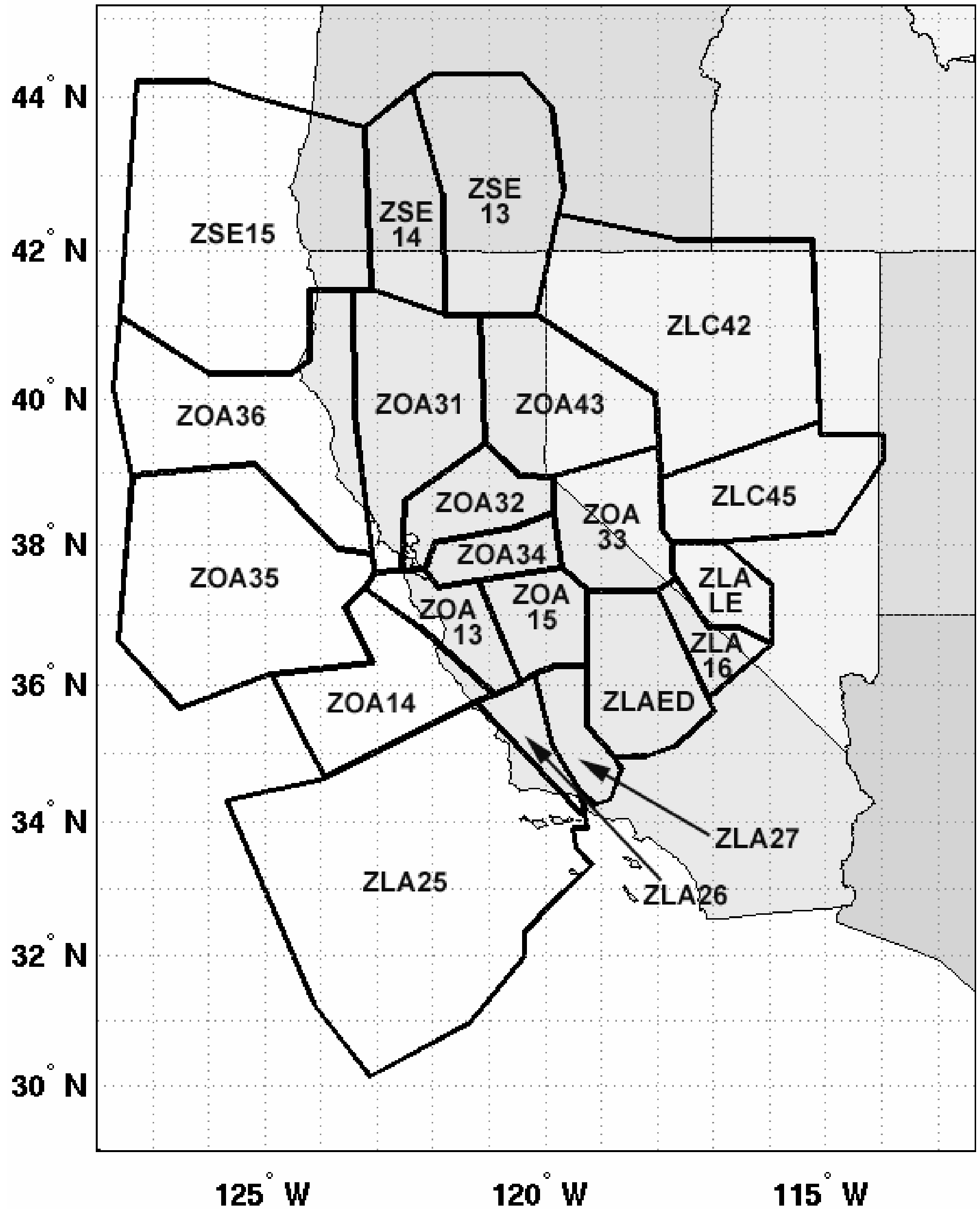
	Eulerian Control volume based	Lagrangian Trajectory based
		
Micro Particles 		
Macro Aggregation of Particles 		

Air Route Traffic Control Centers (ARTCCs)

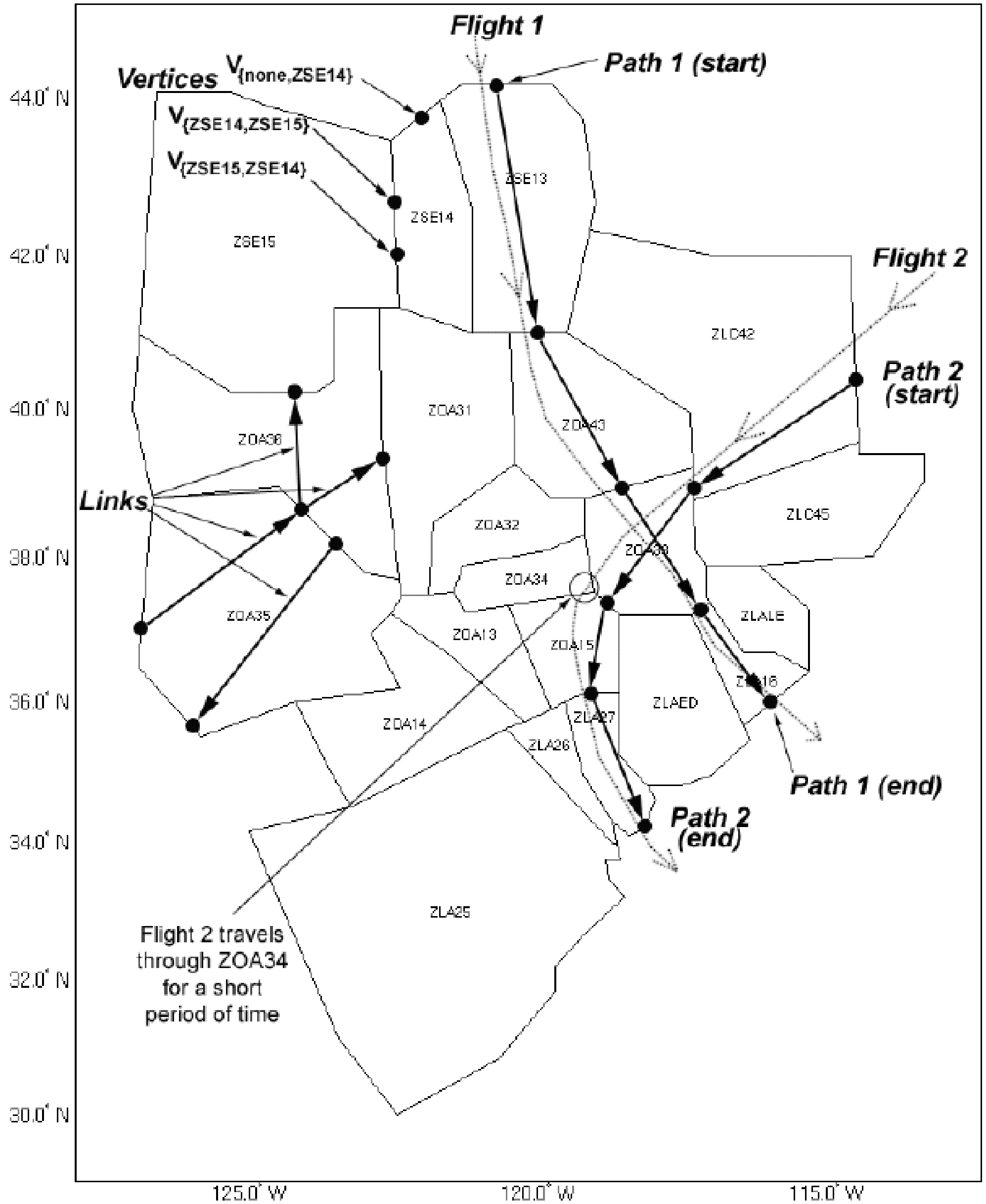


[Sridhar, Soni, Sheth, Chatterji, AIAA GNC 2004]

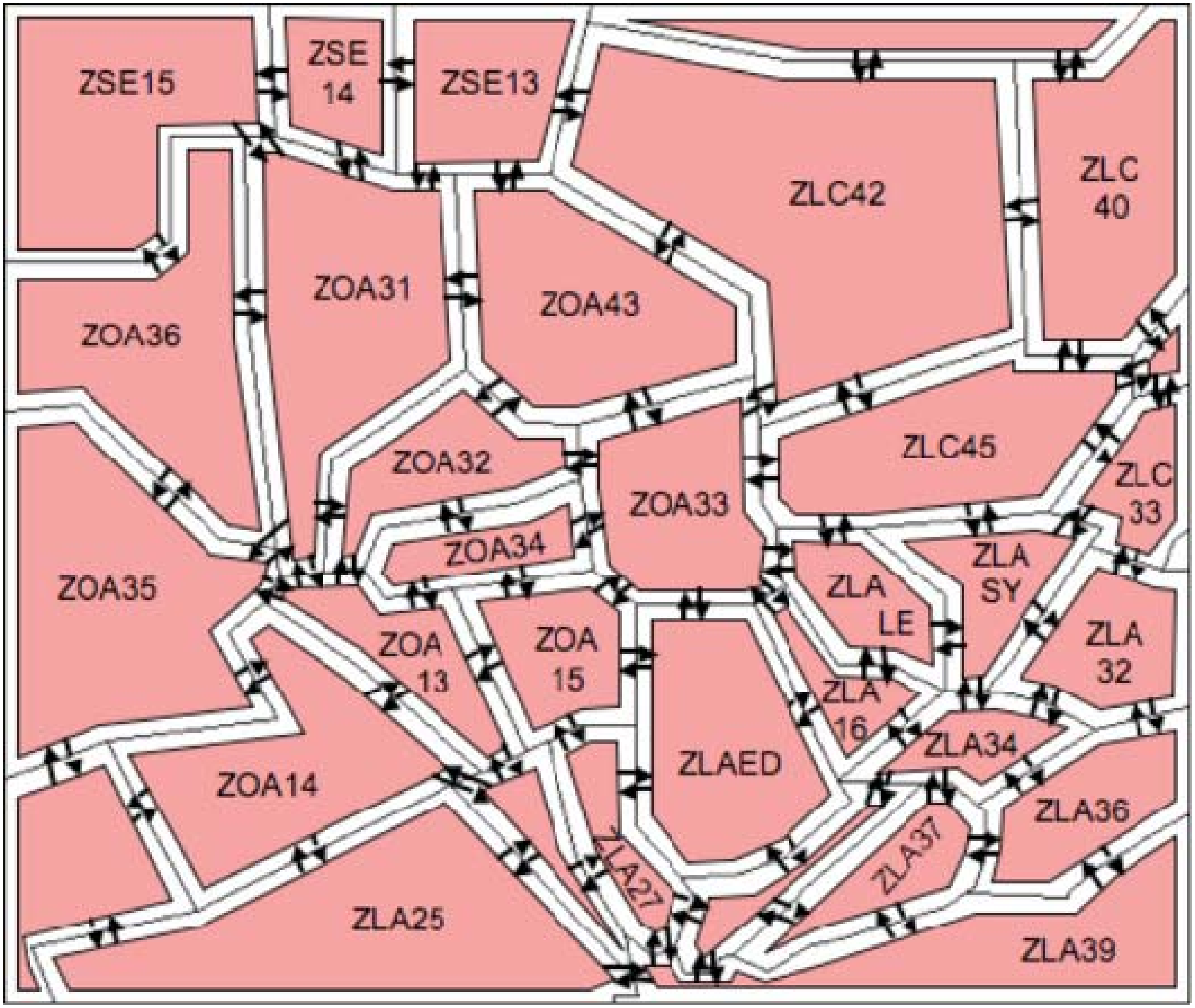
Underlying navigation infrastructure: sectors



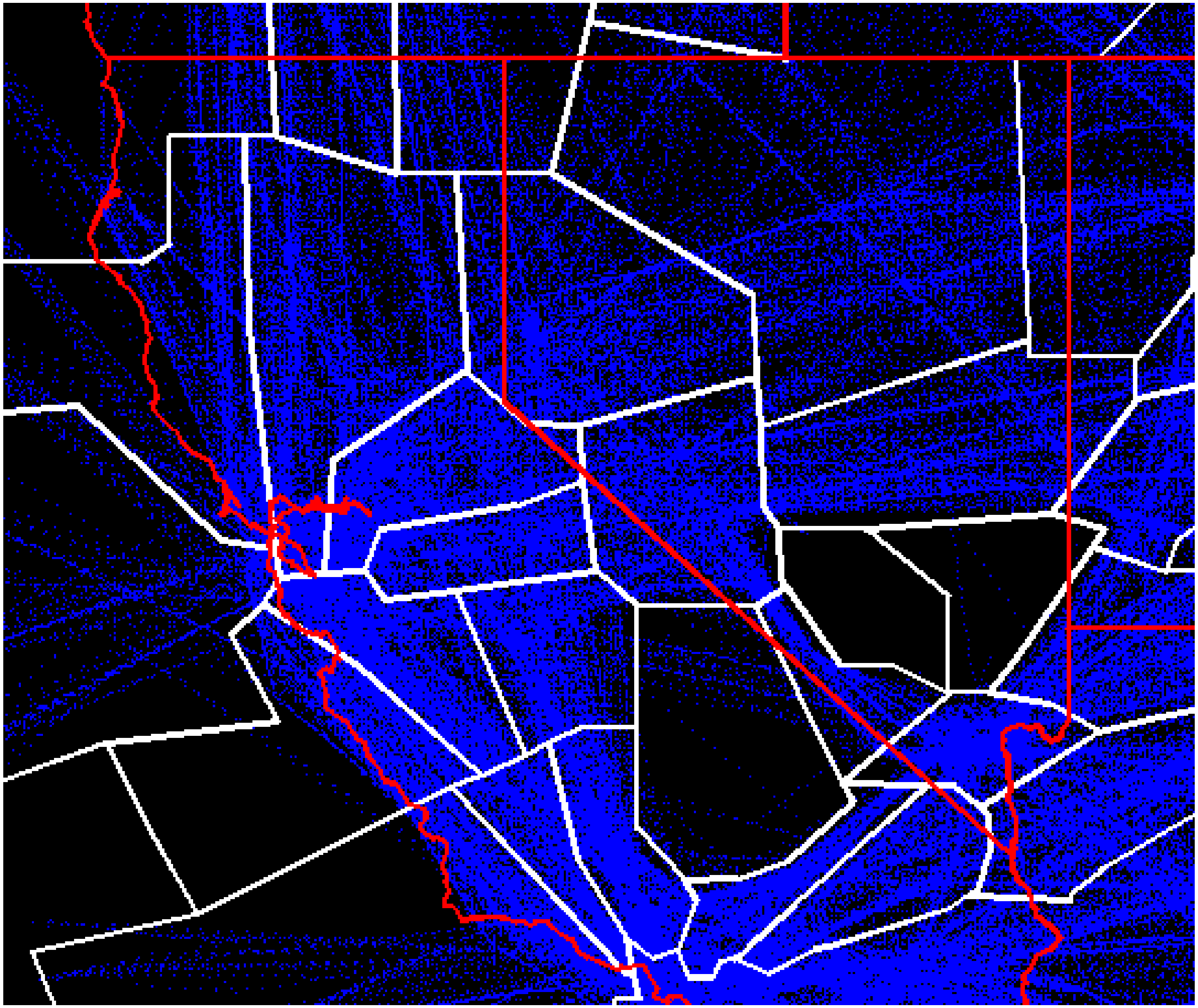
Nature of data used: flows (to be clustered)



Underlying navigation infrastructure: sectors



Conceptual framework: flow aggregation



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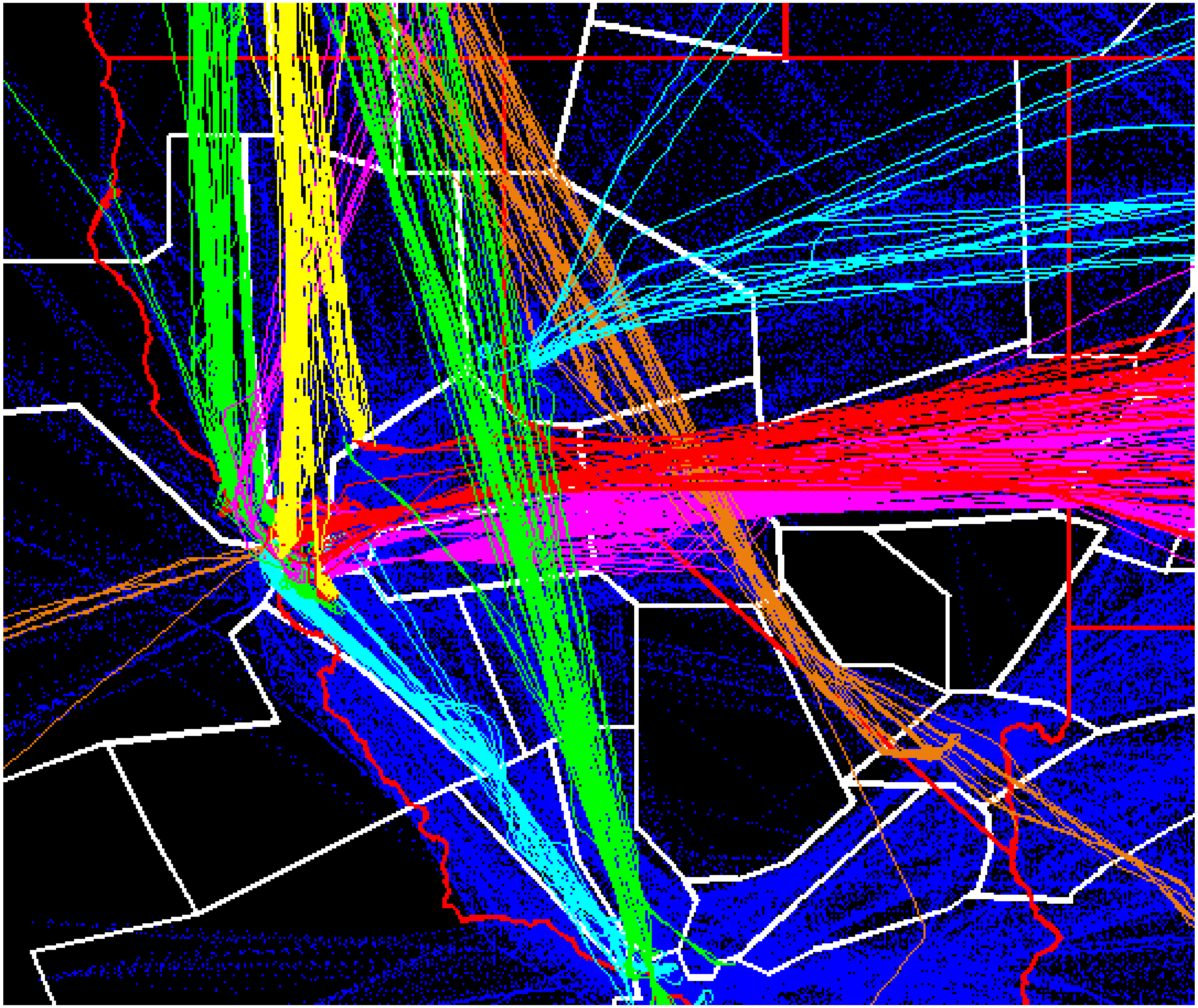
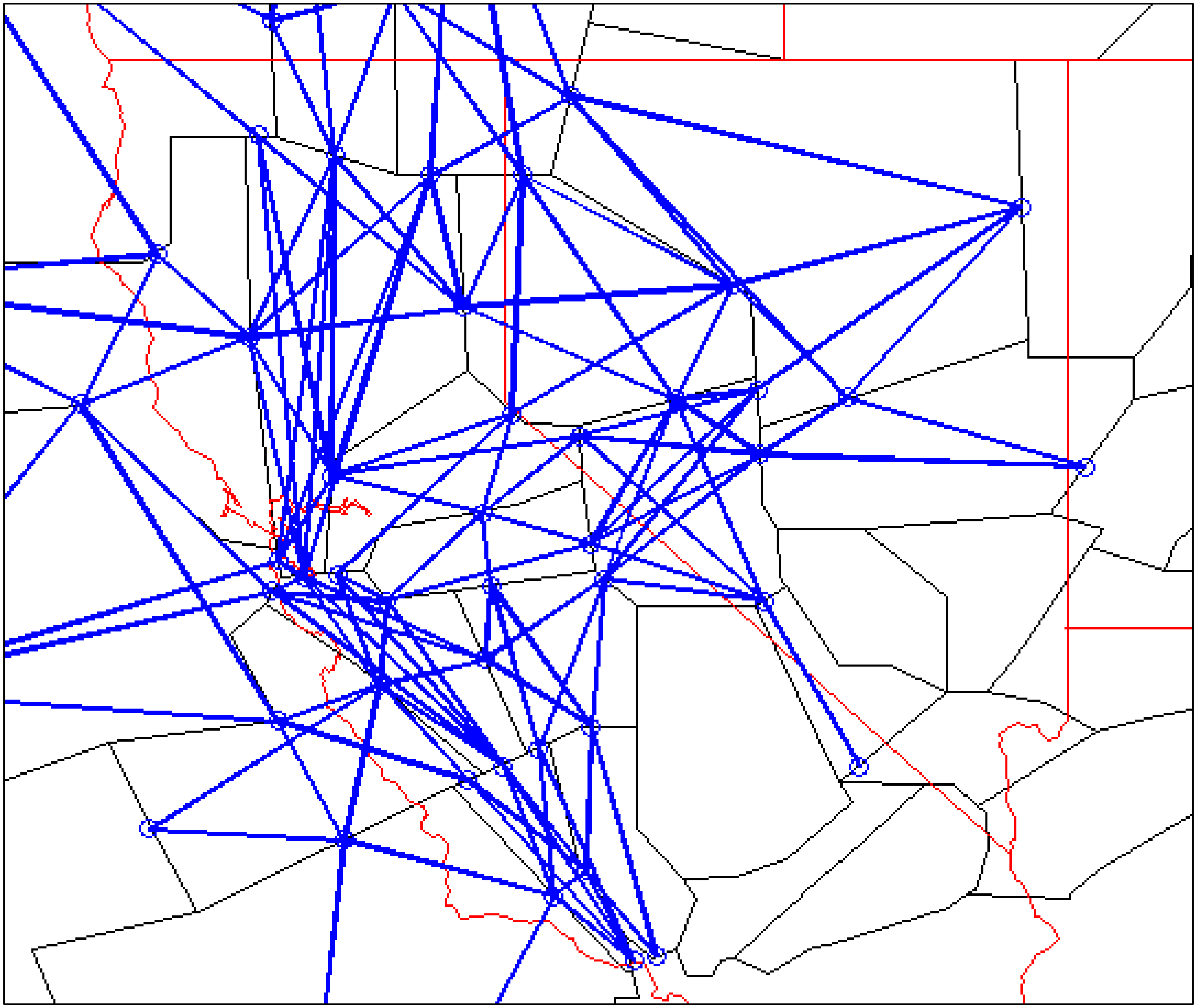
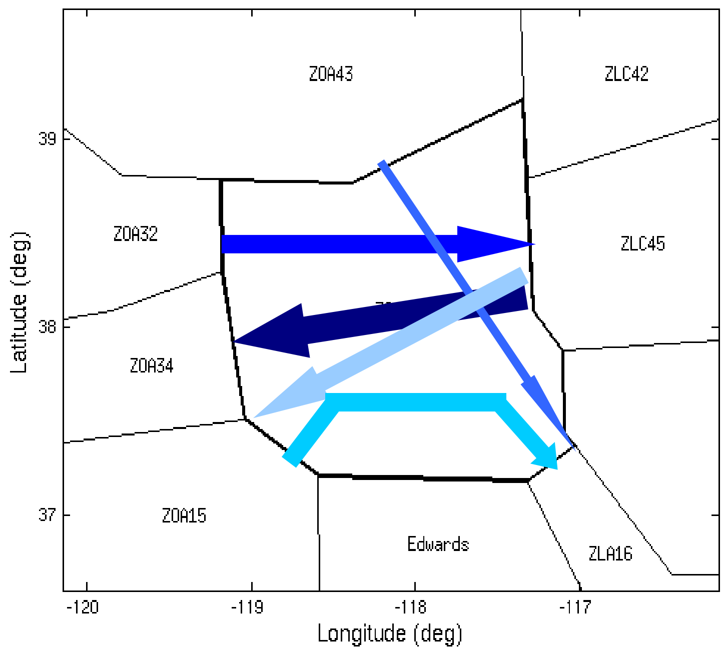
Conceptual framework: flow aggregation

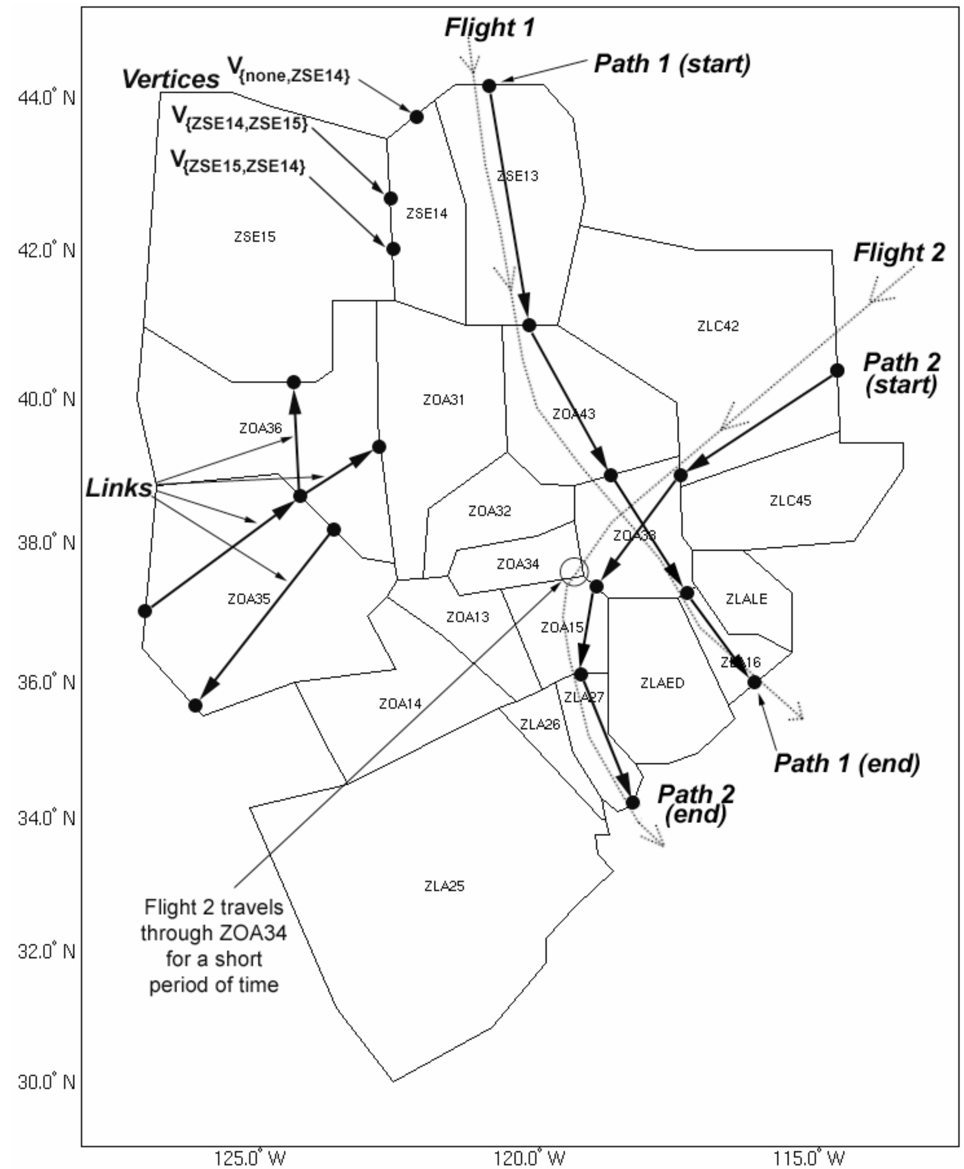


Resulting mathematical object: network



Automated model building





Related work: [Histon, Hansman, 2002]

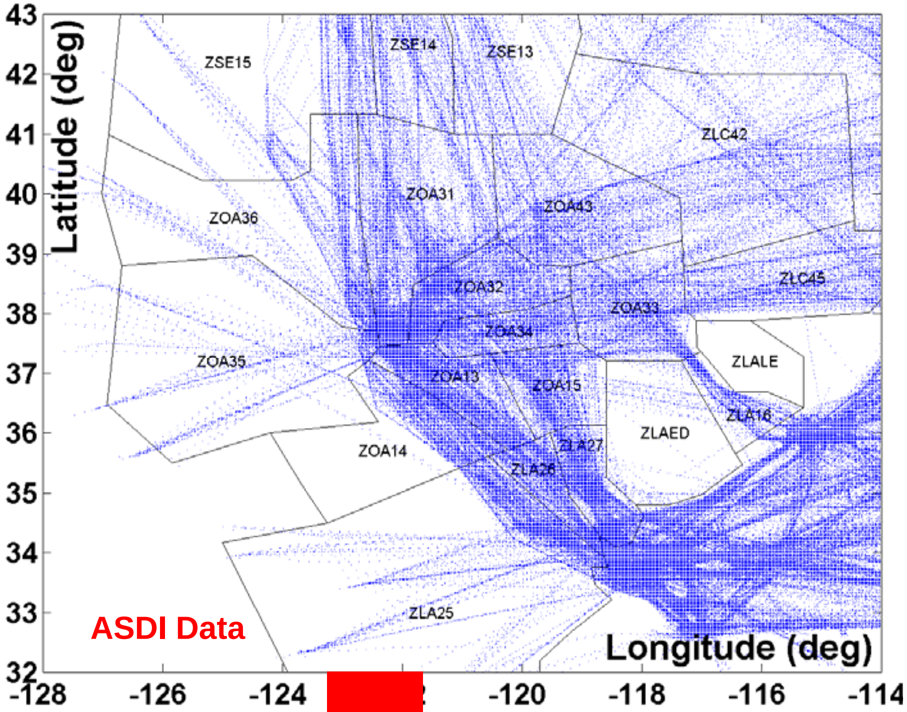
Sequential (automated) algorithm

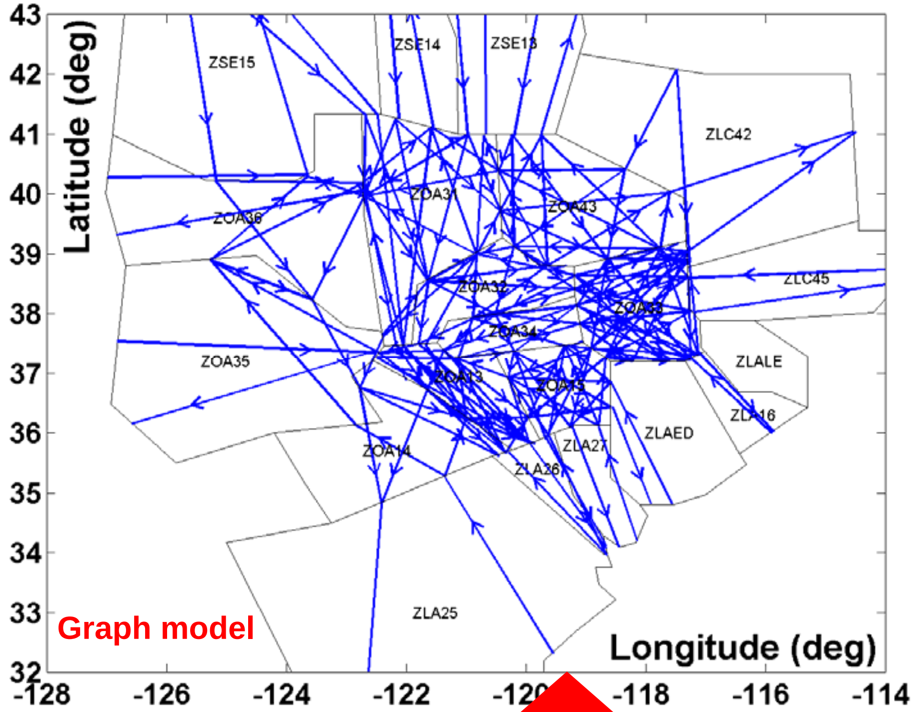
1. Airspace segmentation using sector boundaries
2. Link building using clustering techniques
3. Data aggregation using ASDI/ETMS information (flight plan information)
4. Filtering using LOAs, and observed flow patterns
5. Computation of the aggregate flow pattern features

Output: topology of the flows

Models: graph model of air traffic flow

- Flight tracks – graph theoretical model





automated identification procedure: clustering; pattern recognition

[Robelin et al. 2006]

Network flow graph building algorithm

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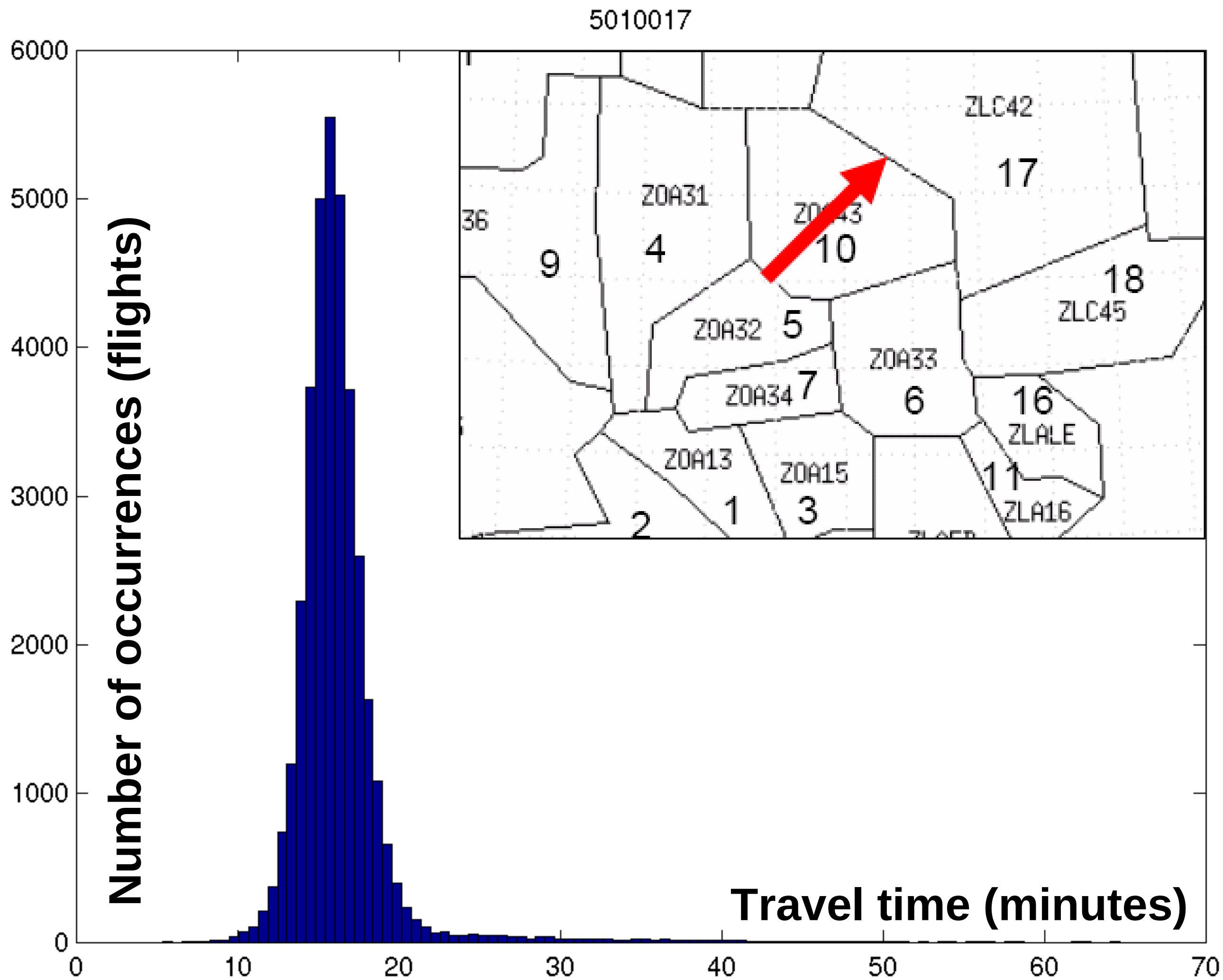
Example of travel time distribution

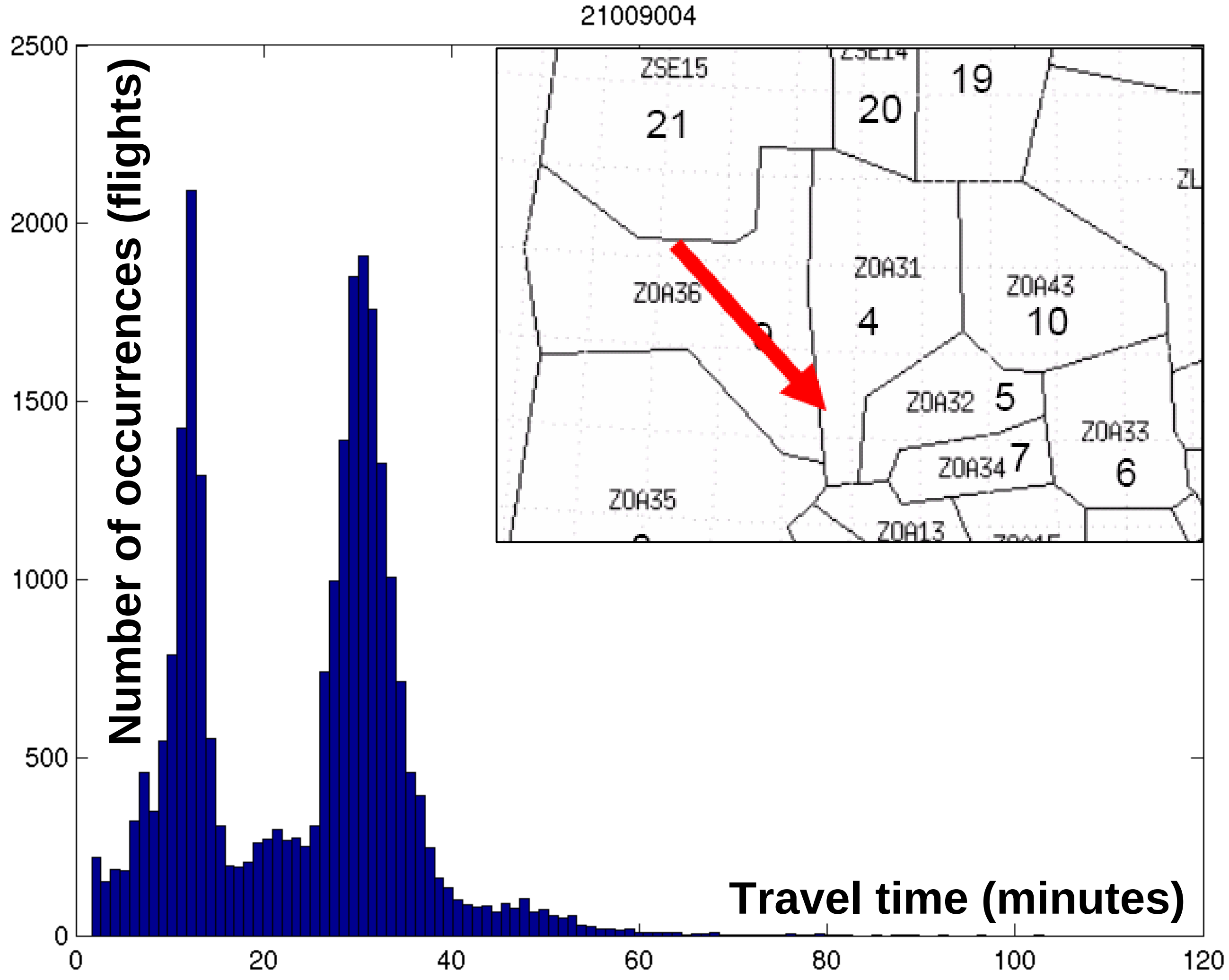


Travel times through the link ZOA32-ZOA43-ZLC42

Departure flights from the bay area.

Related work: [Hoffman, Krozel, Roy, Roth 2003]

Example of "bad" segmentation: second peak



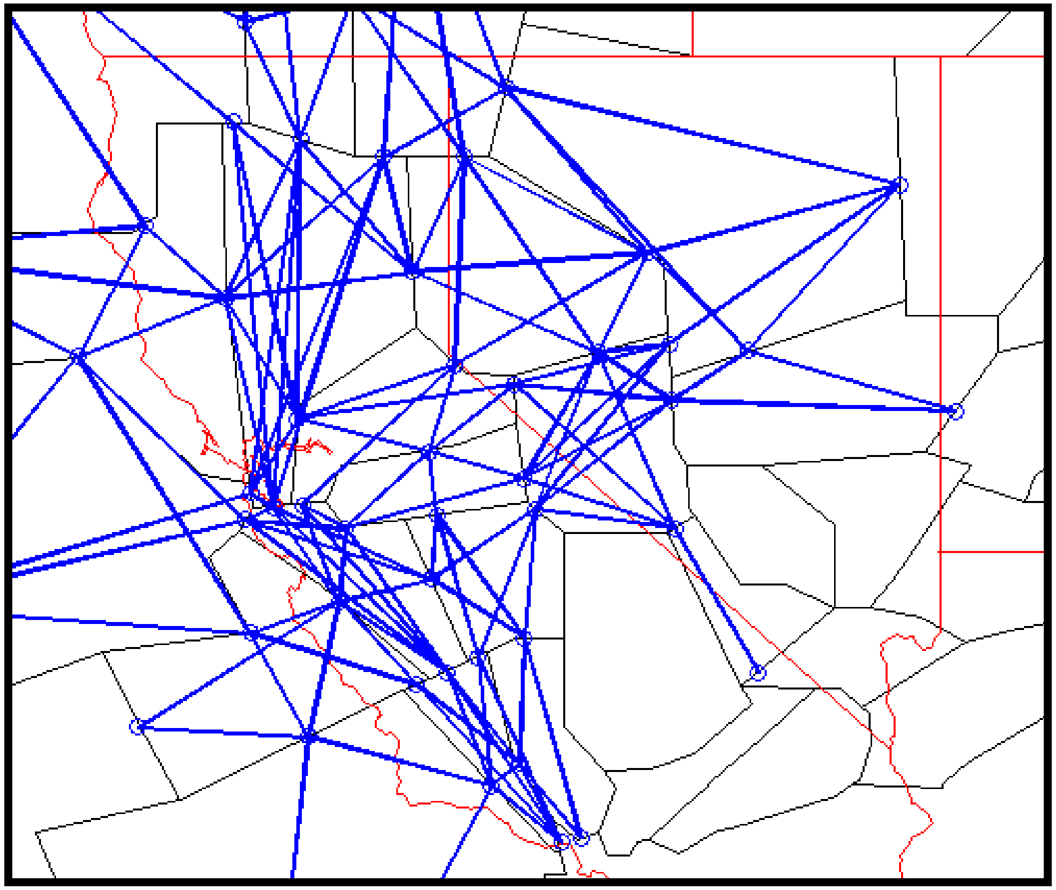
Travel times through the link ZSE15-ZOA26-ZOA31

Possibility: one descending bay area, one passing by

Related work: [Hoffman, Krozel, Roy, Roth 2003]

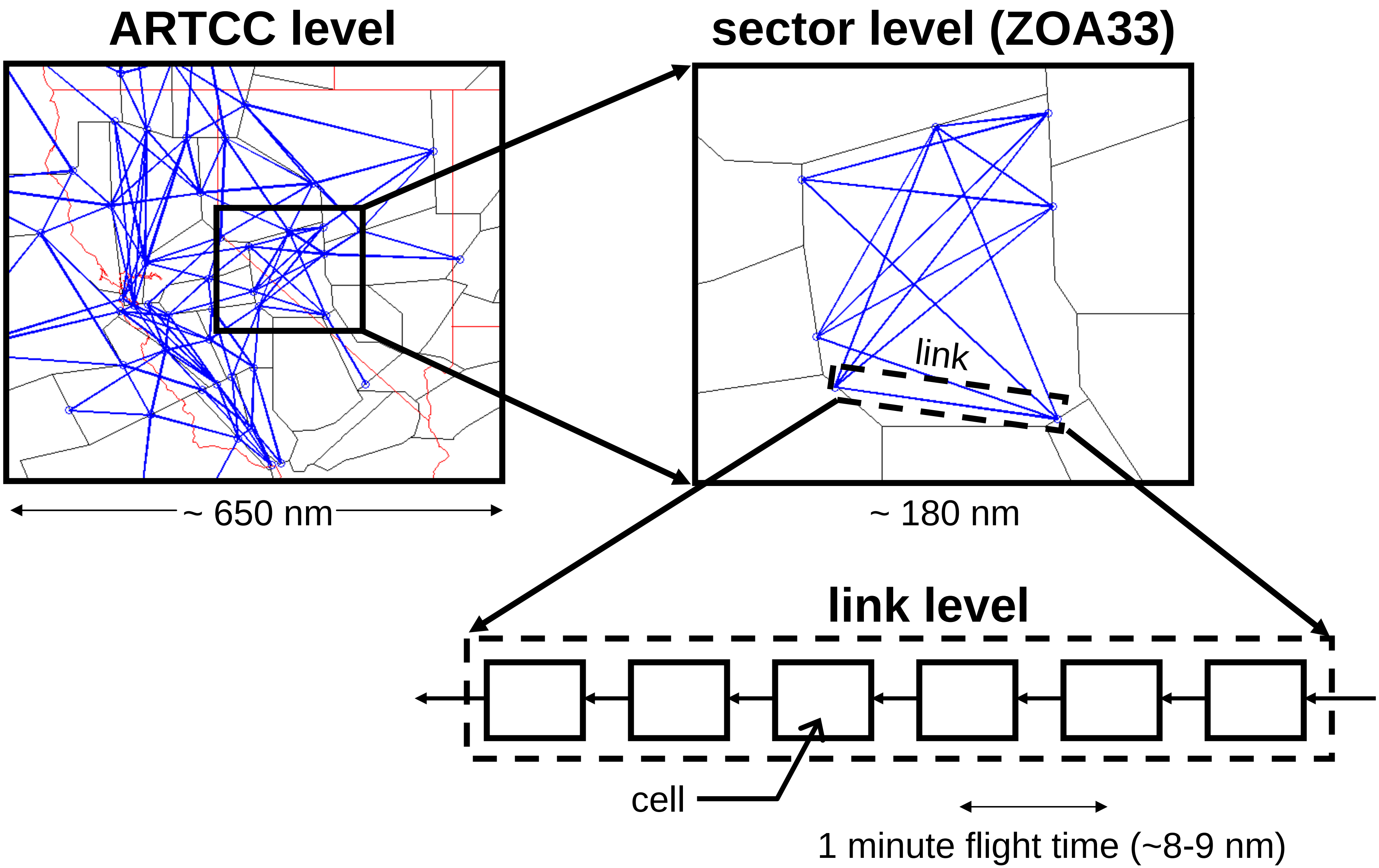
Graph theoretic model

Model of flow patterns for Oakland, and part of Los Angeles, Seattle, and Salt Lake ARTCCs (ZOA, ZLA, ZSE, and ZLC)



Typical directed graph: ~300 links

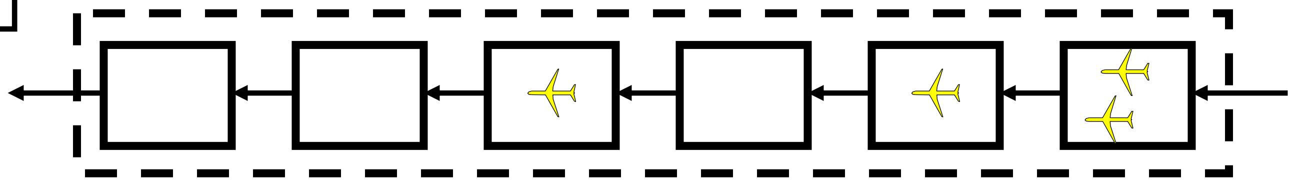
Models #1: Delay Model



Delay Model: Eulerian dynamics on a link

time step
1

delay system at the link level



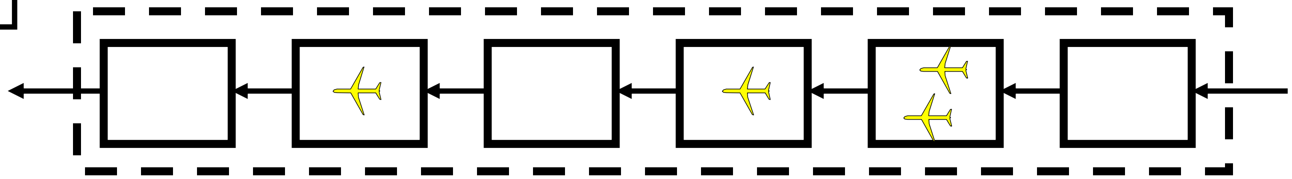
state: cell counts

$$x(1) = \begin{bmatrix} x_6(1) \\ x_5(1) \\ x_4(1) \\ x_3(1) \\ x_2(1) \\ x_1(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

Delay Model: Eulerian dynamics on a link

time step
2

delay system at the link level



state at current time step → $x(2)$; state at previous time step → $x(1)$

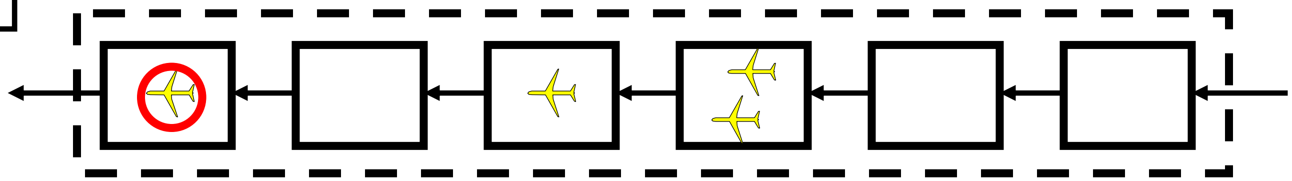
$$\underbrace{\begin{bmatrix}0\\1\\0\\1\\2\\0\end{bmatrix}}_{x(2)} = \begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}\underbrace{\begin{bmatrix}0\\0\\1\\0\\1\\2\end{bmatrix}}_{x(1)} + \begin{bmatrix}1&-1&0&0&0&0\\0&1&-1&0&0&0\\0&0&1&-1&0&0\\0&0&0&1&-1&0\\0&0&0&0&1&-1\\0&0&0&0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}}_{u(1)} + \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}\underbrace{0}_{f(1)}$$

state: cell counts

Delay Model: Eulerian dynamics on a link

time step
3

delay system at the link level

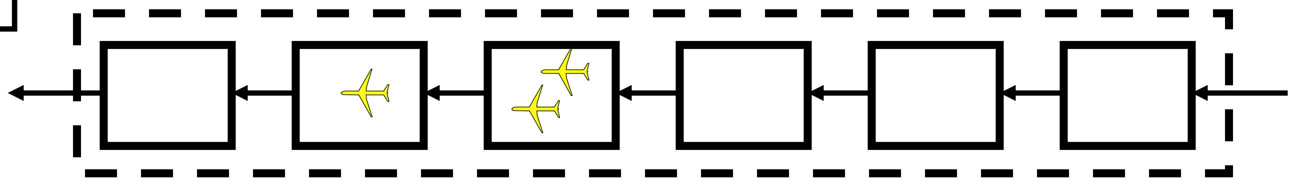


$$\underbrace{\begin{bmatrix}1\\0\\1\\2\\0\\0\end{bmatrix}}_{x(3)} = \begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}\underbrace{\begin{bmatrix}0\\1\\0\\1\\2\\0\end{bmatrix}}_{x(2)} + \begin{bmatrix}1&-1&0&0&0&0\\0&1&-1&0&0&0\\0&0&1&-1&0&0\\0&0&0&1&-1&0\\0&0&0&0&1&-1\\0&0&0&0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}}_{u(2)} + \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}\underbrace{0}_{f(2)}$$

Delay Model: Eulerian dynamics on a link

time step
4

delay system at the link level

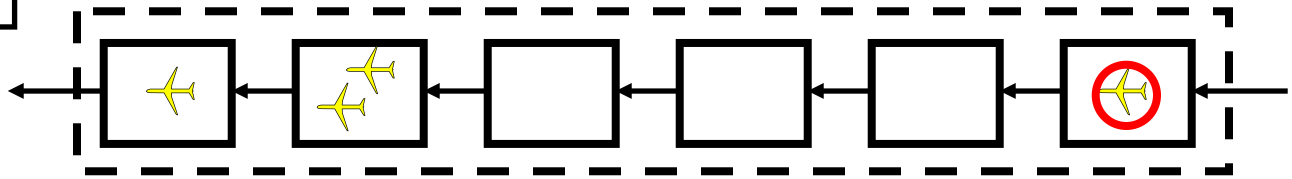


$$\underbrace{\begin{bmatrix}0\\1\\2\\0\\0\\0\end{bmatrix}}_{x(4)} = \begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}\underbrace{\begin{bmatrix}1\\0\\1\\2\\0\\0\end{bmatrix}}_{x(3)} + \begin{bmatrix}1&-1&0&0&0&0\\0&1&-1&0&0&0\\0&0&1&-1&0&0\\0&0&0&1&-1&0\\0&0&0&0&1&-1\\0&0&0&0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}}_{u(3)} + \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}\underbrace{0}_{f(3)}$$

Delay Model: Eulerian dynamics on a link

time step
5

delay system at the link level



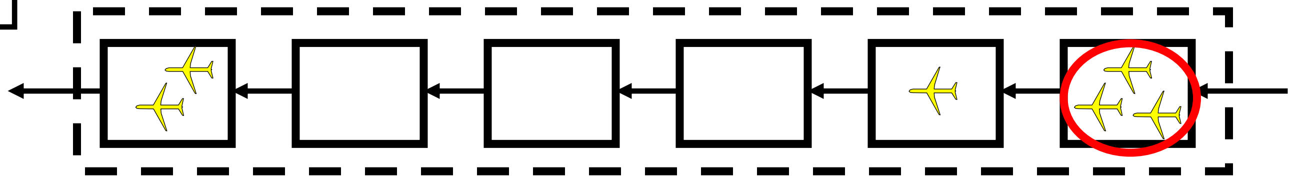
$$\underbrace{\begin{bmatrix}1\\2\\0\\0\\0\\1\end{bmatrix}}_{x(5)} = \begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}\underbrace{\begin{bmatrix}0\\1\\2\\0\\0\\0\end{bmatrix}}_{x(4)} + \begin{bmatrix}1&-1&0&0&0&0\\0&1&-1&0&0&0\\0&0&1&-1&0&0\\0&0&0&1&-1&0\\0&0&0&0&1&-1\\0&0&0&0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}}_{u(4)} + \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}\underbrace{\quad 1 \quad}_{f(4)}$$

input to the link (forcing)

Delay Model: Eulerian dynamics on a link

time step
6

delay system at the link level

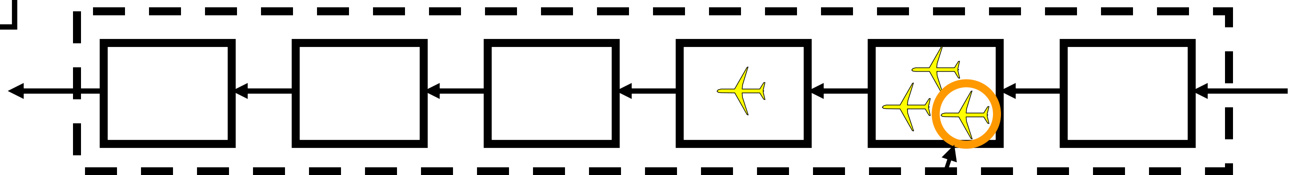


$$\underbrace{\begin{bmatrix}2\\0\\0\\0\\1\\3\end{bmatrix}}_{x(6)} = \begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}\underbrace{\begin{bmatrix}1\\2\\0\\0\\0\\1\end{bmatrix}}_{x(5)} + \begin{bmatrix}1&-1&0&0&0&0\\0&1&-1&0&0&0\\0&0&1&-1&0&0\\0&0&0&1&-1&0\\0&0&0&0&1&-1\\0&0&0&0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}}_{u(5)} + \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}\underbrace{3}_{f(5)}$$

Delay Model: Eulerian dynamics on a link

time step
7

delay system at the link level



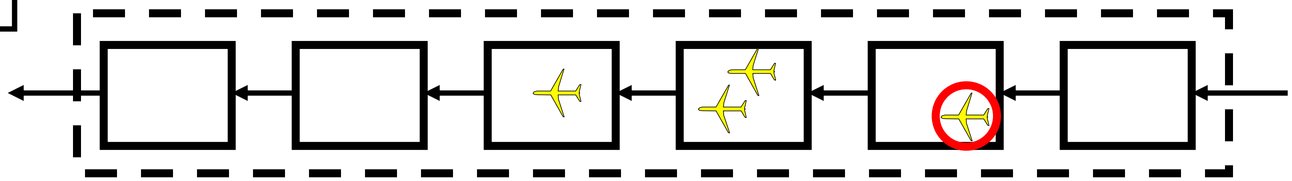
one aircraft about to be delayed

$$\underbrace{\begin{bmatrix}0\\0\\0\\1\\3\\0\end{bmatrix}}_{x(7)} = \begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}\underbrace{\begin{bmatrix}2\\0\\0\\0\\1\\3\end{bmatrix}}_{x(6)} + \begin{bmatrix}1&-1&0&0&0&0\\0&1&-1&0&0&0\\0&0&1&-1&0&0\\0&0&0&1&-1&0\\0&0&0&0&1&-1\\0&0&0&0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}}_{u(6)} + \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}\underbrace{0}_{f(6)}$$

Delay Model: Eulerian dynamics on a link

time step
8

delay system at the link level



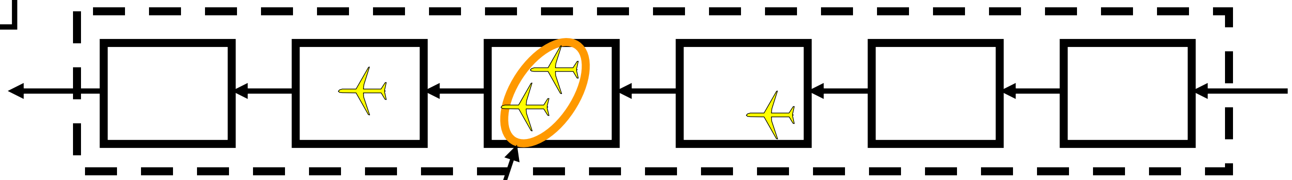
$$\underbrace{\begin{bmatrix}0\\0\\1\\2\\1\\0\end{bmatrix}}_{x(8)} = \begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}\underbrace{\begin{bmatrix}0\\0\\0\\1\\3\\0\end{bmatrix}}_{x(7)} + \begin{bmatrix}1&-1&0&0&0&0\\0&1&-1&0&0&0\\0&0&1&-1&0&0\\0&0&0&1&-1&0\\0&0&0&0&1&-1\\0&0&0&0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}}_{u(7)} + \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}\underbrace{0}_{f(7)}$$

control input

Delay Model: Eulerian dynamics on a link

time step
9

delay system at the link level



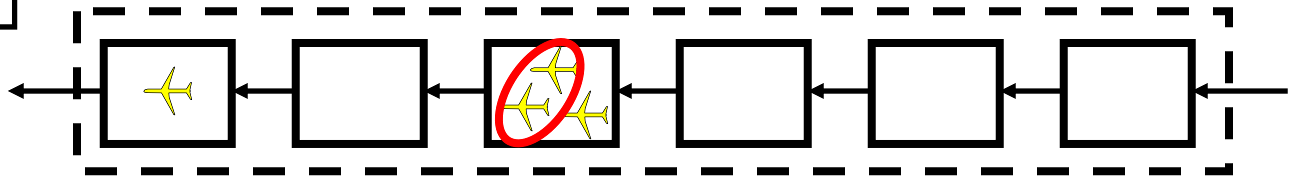
two aircraft about to be delayed

$$\underbrace{\begin{bmatrix}0\\1\\2\\1\\0\\0\end{bmatrix}}_{x(9)} = \begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}\underbrace{\begin{bmatrix}0\\0\\1\\2\\1\\0\end{bmatrix}}_{x(8)} + \begin{bmatrix}1&-1&0&0&0&0\\0&1&-1&0&0&0\\0&0&1&-1&0&0\\0&0&0&1&-1&0\\0&0&0&0&1&-1\\0&0&0&0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}}_{u(8)} + \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}\underbrace{0}_{f(8)}$$

Delay Model: Eulerian dynamics on a link

time step
10

delay system at the link level



$$\underbrace{\begin{bmatrix}1\\0\\3\\0\\0\\0\end{bmatrix}}_{x(10)} = \begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}\underbrace{\begin{bmatrix}0\\1\\2\\1\\9\\0\end{bmatrix}}_{x(9)} + \begin{bmatrix}1&-1&0&0&0&0\\0&1&-1&0&0&0\\0&0&1&-1&0&0\\0&0&0&1&-1&0\\0&0&0&0&1&-1\\0&0&0&0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}0\\0\\2\\0\\0\\0\end{bmatrix}}_{u(9)} + \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}\underbrace{0}_{f(9)}$$

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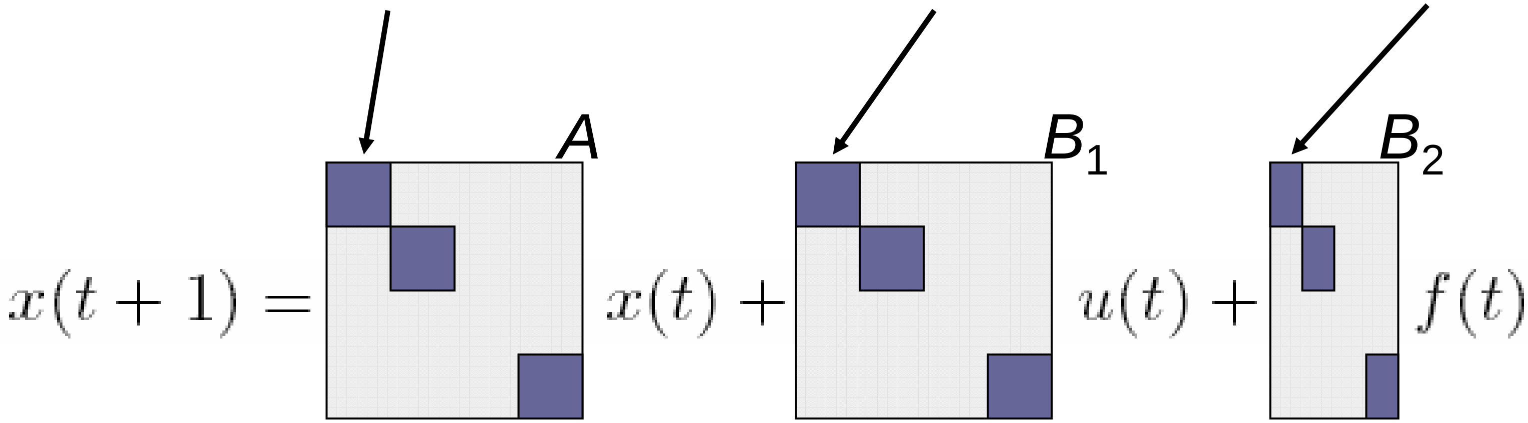
Models #1: Delay Model

link level

$$\begin{bmatrix} x_6(t+1) \\ x_5(t+1) \\ x_4(t+1) \\ x_3(t+1) \\ x_2(t+1) \\ x_1(t+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A_i} \begin{bmatrix} x_6(t) \\ x_5(t) \\ x_4(t) \\ x_3(t) \\ x_2(t) \\ x_1(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{B_{1,i}} \begin{bmatrix} u_6(t) \\ u_5(t) \\ u_4(t) \\ u_3(t) \\ u_2(t) \\ u_1(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{B_{2,i}} f(t)$$



Air Route Traffic Control Center (ARTCC) level

Sparse LTI dynamical system:
blocks are nilpotent or upper diagonal matrices

Control theoretic model of air traffic flow

System can be described by a Linear Dynamical System

- LTI system is a good first order approximation of the system
- For the entire NAS
- Difference with the seminal work of Bertsimas/Stock: model (vs. paradigm)

Current size processed for the model (from ASDI/ETMS data)

- 20 ARTCCs in US (continental)
- 284 sectors
- 9572 links
- ~300,000 actual Paths
- 110,081 OD pairs

Graph theoretical model

- 20 ARTCCs in US (continental)
- 284 sectors
- 9572 links
- 110,081 OD pairs

Naive approach of the LTI system

Observability and controllability matrices can be built

$$\mathcal{C}_t = \begin{bmatrix} B & AB & \cdots & A^{t-1}B \end{bmatrix}$$

$$\mathcal{O}_t = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t-1} \end{bmatrix}$$

Matrices are full rank! System is observable and controllable!

Obviously this does not work because

1) System is integer valued
2) Even if the system is relaxed, the system is constrained:
    - For the control
    - For the state

However, constrained linear optimization works

Standard constrained optimization provides computationally tractable alternatives to classical control techniques, which are unfortunately inapplicable.

Example: global delay mitigation

$$\textbf{min:} \quad \sum_{k=0}^{N} c^T x_k$$

subject to:

$$x_0 = B_2 f_0$$

$$x_{k+1} = A x_k + B_1 u_k + B_2 f_k, \; k \in \{0, \cdots, N-1\}$$

$$E x_k + L u_k \leq M, \; k \in \{0, \cdots, N-1\}$$

$$x_N \in \chi_f$$

N: number of time steps, *c*: vector of 1's

E, L, M: implement user-specified constraints (capacity, nonnegativity, etc)

χ_f: set of feasible final states, x, f, u, A, B_1, B_2: as defined earlier

MILP formulation of control

min: $\displaystyle\sum_{k=0}^{N} c^T x_k$ travel time

subject to:

$$x_0 = B_2 f_0$$
$$x_{k+1} = A x_k + B_1 u_k + B_2 f_k,\ k \in \{0, \cdots, N-1\}$$
$$E x_k + L u_k \leq M,\ k \in \{0, \cdots, N-1\}$$
$$x_N \in \chi_f$$

N: number of time steps
c: vector of 1's
E, L, M: implement user-specified constraints (capacity, nonnegativity, etc)
χ_f: set of feasible final states
x, *f*, *u*, *A*, B_1, B_2: as defined earlier

MILP formulation of control

min: $$\sum_{k=0}^{N} c^T x_k$$

subject to:

$$x_0 = B_2 f_0$$

$$x_{k+1} = Ax_k + B_1 u_k + B_2 f_k,\ k \in \{0, \cdots, N-1\}$$

$$Ex_k + Lu_k \leq M,\ k \in \{0, \cdots, N-1\}$$

$$x_N \in \chi_f$$

initial condition

N: number of time steps

c: vector of 1's

E, L, M: implement user-specified constraints (capacity, nonnegativity, etc)

χ_f: set of feasible final states

x, *f*, *u*, *A*, B_1, B_2: as defined earlier

MILP formulation of control

min: $$\sum_{k=0}^{N} c^T x_k$$

subject to:

$$x_0 = B_2 f_0$$

$$x_{k+1} = A x_k + B_1 u_k + B_2 f_k,\ k \in \{0, \cdots, N-1\}$$

$$E x_k + L u_k \leq M,\ k \in \{0, \cdots, N-1\}$$

$$x_N \in \chi_f$$

dynamics

N: number of time steps

c: vector of 1's

E, L, M: implement user-specified constraints (capacity, nonnegativity, etc)

χ_f: set of feasible final states

x, *f*, *u*, *A*, B_1, B_2: as defined earlier

MILP formulation of control

min: $$\sum_{k=0}^{N} c^T x_k$$

subject to:

$$x_0 = B_2 f_0$$

$$x_{k+1} = Ax_k + B_1 u_k + B_2 f_k,\ k \in \{0, \cdots, N-1\}$$

$$Ex_k + Lu_k \leq M,\ k \in \{0, \cdots, N-1\}$$

$$x_N \in \chi_f$$

N: number of time steps
c: vector of 1's
E, L, M: implement user-specified constraints (capacity, nonnegativity, etc)
χ_f: set of feasible final states
x, f, u, A, B_1, B_2: as defined earlier

user-specified constraints

- sector capacity
- nonnegativity of *x* and *u*
- *u* less than or equal to *x*

MILP formulation of control

min: $$\sum_{k=0}^{N} c^T x_k$$

subject to:

$$x_0 = B_2 f_0$$

$$x_{k+1} = A x_k + B_1 u_k + B_2 f_k, \; k \in \{0, \cdots, N-1\}$$

$$E x_k + L u_k \leq M, \; k \in \{0, \cdots, N-1\}$$

$$x_N \in \chi_f$$

final state

N: number of time steps

c: vector of 1's

E, L, M: implement user-specified constraints (capacity, nonnegativity, etc)

χ_f: set of feasible final states

x, f, u, A, B_1, B_2: as defined earlier

MILP formulation of control

MILP formulation

min:

$$\sum_{k=0}^{N} c^T x_k$$

subject to:

$$x_0 = B_2 f_0$$

$$x_{k+1} = Ax_k + B_1 u_k + B_2 f_k,\ k \in \{0, \cdots, N-1\}$$

$$Ex_k + Lu_k \leq M,\ k \in \{0, \cdots, N-1\}$$

$$x_N \in \chi_f$$

Challenges:
>1M variables,
>1M constraints

N: number of time steps
c: vector of 1's
E, L, M: implement user-specified constraints (capacity, nonnegativity, etc)
χ_f: set of feasible final states
x, f, u, A, B_1, B_2: as defined earlier

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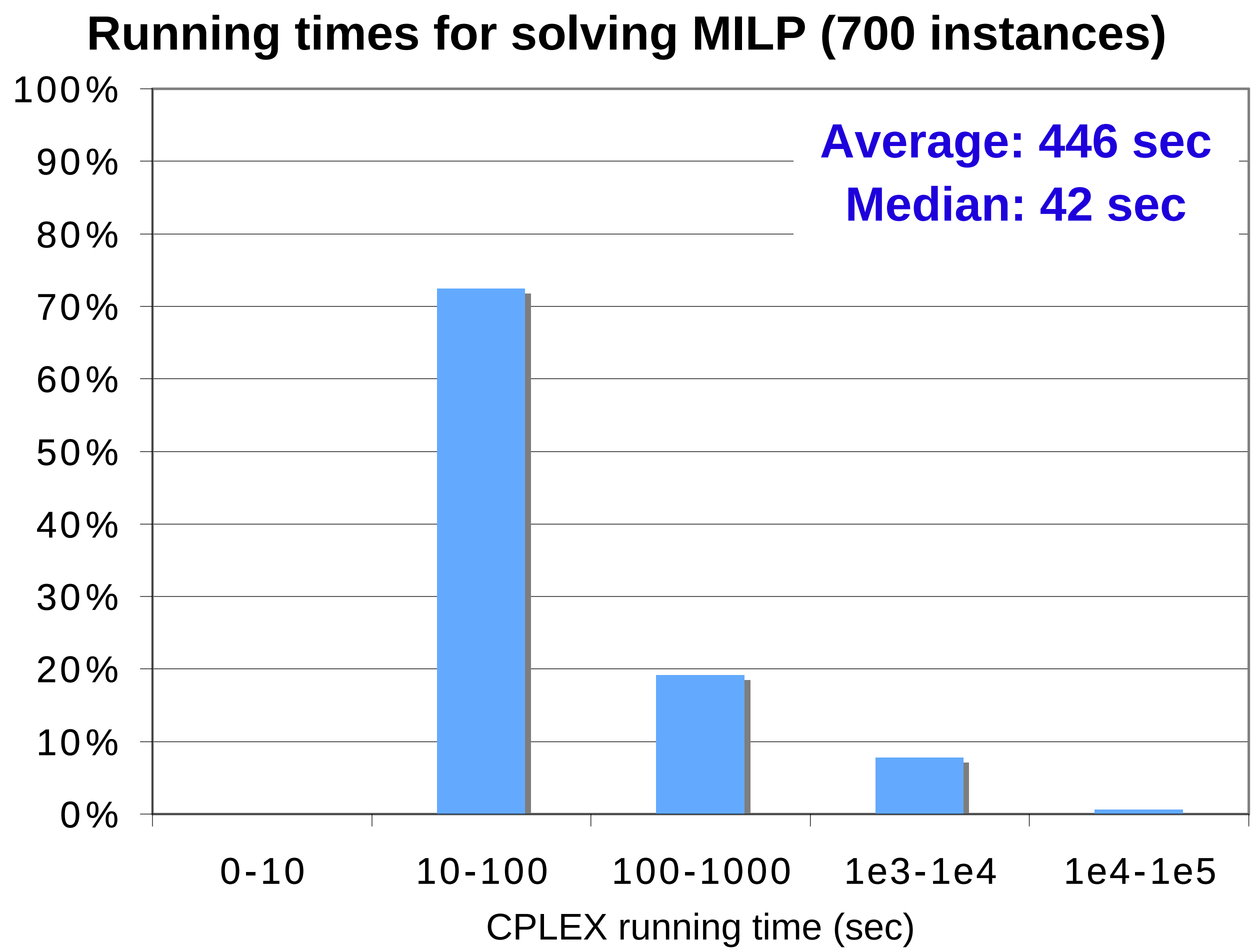
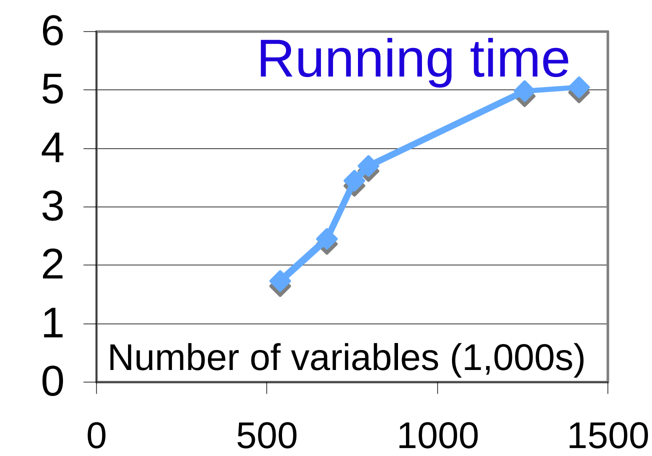
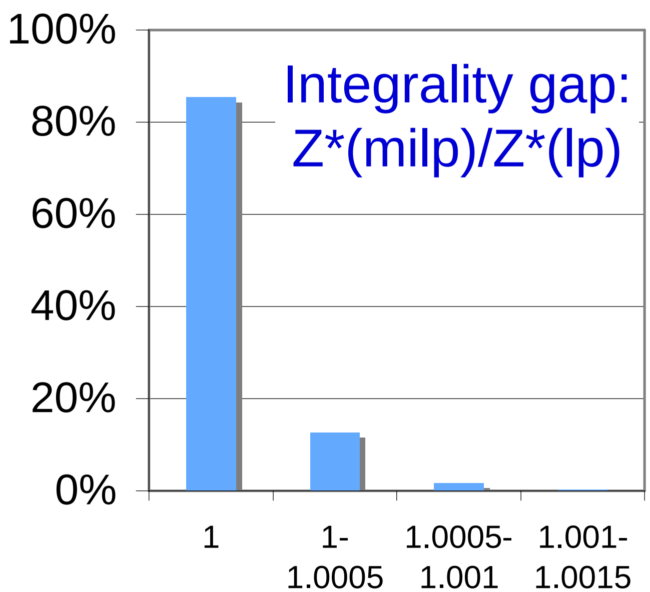
MILP running time



MILP: large variance of CPLEX times ➡ LP relaxation

LP relaxation results





MILP ➡ LP:

Standard deviation on running time divided by 4

LP results very close to MILP results (only a few variables different)

Example of MILP control

MILP control of aggregate Eulerian network airspace models

ATC actuation to control aircraft counts

Charles-Antoine Robelin, Dengfeng Sun, Guoyuan Wu, and Alexandre Bayen

Berkeley
University of California

NASA

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Going beyond LP, IP, MILP…

Posed as a general IP (relaxed to a LP), the problem becomes computationally challenging for the whole NAS if no care is taken of the structure of the problem (limits of the “blind” LP approach).

Problem is in fact a multicommodity flow problem

- Static version is known to be NP-hard
- Relaxation of the IP is a LP
- LP relaxation can be solved easily with dual decomposition
- opt(LP) = opt(LP*)

Lagrangian decomposition

$$\min_{u} \quad \sum_{t=1}^{N} c^T x_t$$

subject to

$$x_0 = B_2 f_0$$
$$x_{t+1} = A x_t + B_1 u_t + B_2 f_t, t = 0, \cdots, N-1$$
$$\sum_{i \in Q_n} x_t^i \leq C_n, n = 1, \cdots, |S|, t = 1, \cdots, N.$$

$$\min_{u} \quad \sum_{t=1}^{N} \left(\sum_{k=1}^{K} c^{k^T} x_t^k \right)$$

subject to

$$x_0^k = B_2^k f_0^k, k = 1, \cdots, K$$
$$x_{t+1}^k = A^k x_t^k + B_1^k u_t^k + B_2^k f_t^k, t = 0, \cdots, N-1, k = 1, \cdots, K$$
$$\sum_{k=1}^{K} \sum_{i \in Q_n} x_t^{i,k} \leq C_n, n = 1, \cdots, |S|.$$

Lagrangian decomposition

$$\min_{u, C_n^k, k=1,\cdots,K, n=1,\cdots,|S|} \sum_{k=1}^{K} \left(\sum_{t=1}^{N} {c^k}^T x_t^k \right)$$

subject to

$$x_0^k = B_2^k f_0^k, k = 1, \cdots, K$$

$$x_{t+1}^k = A^k x_t^k + B_1^k u_t^k + B_2^k f_t^k, t = 0, \cdots, N-1, k = 1, \cdots, K$$

$$\sum_{i \in Q_n} x_t^{i,k} = C_n^k, n = 1, \cdots, |S|, k = 1, \cdots, K.$$

$$\sum_{k=1}^{K} C_n^k \leq C_n, n = 1, \cdots, |S|.$$

$$p^* := \min_{u, C_n^k, k=1,\cdots,K, n=1,\cdots,|S|} \max_{\lambda \geq 0} \sum_{k=1}^{K} \left(\sum_{t=1}^{N} {c^k}^T x_t^k \right) + \sum_{n=1}^{|S|} \left[\lambda_n \left(\sum_{k=1}^{K} C_n^k - C_n \right) \right]$$

subject to

$$x_0^k = B_2^k f_0^k, k = 1, \cdots, K$$

$$x_{t+1}^k = A^k x_t^k + B_1^k u_t^k + B_2^k f_t^k, t = 0, \cdots, N-1, k = 1, \cdots, K$$

$$\sum_{i \in Q_n} x_t^{i,k} = C_n^k, n = 1, \cdots, |S|, k = 1, \cdots, K.$$

Lagrangian decomposition

$$p^* := \min_{u, C_n^k, k=1,\cdots,K, n=1,\cdots,|S|} \max_{\lambda \geq 0} \sum_{k=1}^{K} \left(\sum_{t=1}^{N} c^{k^T} x_t^k \right) + \sum_{n=1}^{|S|} \left[\lambda_n \left(\sum_{k=1}^{K} C_n^k - C_n \right) \right]$$

subject to

$$x_0^k = B_2^k f_0^k, k = 1, \cdots, K$$
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$$\sum_{i \in Q_n} x_t^{i,k} = C_n^k, n = 1, \cdots, |S|, k = 1, \cdots, K.$$

$$d^* := \max_{\lambda \geq 0} \min_{u, C_n^k, k=1,\cdots,K, n=1,\cdots,|S|} \sum_{k=1}^{K} \left(\sum_{t=1}^{N} c^{k^T} x_t^k \right) + \sum_{n=1}^{|S|} \left[\lambda_n \left(\sum_{k=1}^{K} C_n^k - C_n \right) \right]$$

subject to

$$x_0^k = B_2^k f_0^k, k = 1, \cdots, K$$
$$x_{t+1}^k = A^k x_t^k + B_1^k u_t^k + B_2^k f_t^k, t = 0, \cdots, N-1, k = 1, \cdots, K$$
$$\sum_{i \in Q_n} x_t^{i,k} = C_n^k, n = 1, \cdots, |S|, k = 1, \cdots, K.$$

Lagrangian decomposition

$$\begin{aligned}
d^* &= \max_{\lambda\geq 0} \min_{u, C_n^k, k=1,\cdots,K, n=1,\cdots,|S|} \sum_{k=1}^{K}\left(\sum_{t=1}^{N} c^{k^T} x_t^k\right) + \sum_{n=1}^{|S|}\left[\lambda_n\left(\sum_{k=1}^{K} C_n^k - C_n\right)\right] \\
&= \max_{\lambda\geq 0} \min_{u, C_n^k, k=1,\cdots,K, n=1,\cdots,|S|} \sum_{k=1}^{K}\left(\sum_{t=1}^{N} c^{k^T} x_t^k\right) + \sum_{n=1}^{|S|}\left(\sum_{k=1}^{K} \lambda_n C_n^k - \lambda_n C_n\right) \\
&= \max_{\lambda\geq 0} \min_{u, C_n^k, k=1,\cdots,K, n=1,\cdots,|S|} \sum_{k=1}^{K}\left(\sum_{t=1}^{N} c^{k^T} x_t^k\right) + \sum_{n=1}^{|S|}\sum_{k=1}^{K} \lambda_n C_n^k - \sum_{n=1}^{|S|} \lambda_n C_n \\
&= \max_{\lambda\geq 0}\left\{-\sum_{n=1}^{|S|} \lambda_n C_n + \min_{u, C_n^k, k=1,\cdots,K, n=1,\cdots,|S|} \sum_{k=1}^{K}\left(\sum_{t=1}^{N} c^{k^T} x_t^k\right) + \sum_{k=1}^{K}\sum_{n=1}^{|S|} \lambda_n C_n^k\right\} \\
&= \max_{\lambda\geq 0}\left\{-\sum_{n=1}^{|S|} \lambda_n C_n + \min_{u, C_n^k, k=1,\cdots,K, n=1,\cdots,|S|} \sum_{k=1}^{K}\left(\sum_{t=1}^{N} c^{k^T} x_t^k + \sum_{n=1}^{|S|} \lambda_n C_n^k\right)\right\} \\
&= \max_{\lambda\geq 0}\left\{-\sum_{n=1}^{|S|} \lambda_n C_n + \sum_{k=1}^{K}\left(\min_{u, C_n^k, n=1,\cdots,|S|} \sum_{t=1}^{N} c^{k^T} x_t^k + \sum_{n=1}^{|S|} \lambda_n C_n^k\right)\right\} \\
&= \max_{\lambda\geq 0}\left\{-\sum_{n=1}^{|S|} \lambda_n C_n + \sum_{k=1}^{K} d^{k^*}(\lambda)\right\}
\end{aligned}$$

where

$$d^{k^*}(\lambda) = \min_{u, C_n^k, n=1,\cdots,|S|} \sum_{t=1}^{N} c^{k^T} x_t^k + \sum_{n=1}^{|S|} \lambda_n C_n^k$$

Lagrangian decomposition

Sub-problems (very tractable)

$$d^{k*}(\lambda) = \min_{u, C_n^k, n=1,\cdots,|S|} \sum_{t=1}^{N} {c^k}^T x_t^k + \sum_{n=1}^{|S|} \lambda_n C_n^k$$

subject to

$$x_0^k = B_2^k f_0^k$$

$$x_{t+1}^k = A^k x_t^k + B_1^k u_t^k + B_2^k f_t^k, t = 0, \cdots, N-1$$

$$\sum_{i \in Q_n} x_t^{i,k} = C_n^k, n = 1, \cdots, |S|.$$

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Future directions

1. Optimization
    1. Dual decomposition
    2. Multi commodity flow (dynamic): combinatorial optimization algorithms
2. Scenarios of interest
    1. En route delay mitigation (MILP → multicommodity flow)
    2. Delay multiplier minimization (NEXTOR collaboration)
    3. Dynamic sectorization of the en route airspace
3. Computational issues
    1. Comparison of computational performance of various algorithms
    2. Convergence acceleration
    3. Link with operational tools (FACET in particular)