PDE Methods for Image Processing

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Final Project for ME236/EE291/CE291F

Outline

- •Traditional Methods in Image Processing
- •Mathematical Axioms for Image transformation
- •PDEs as solutions to these Axioms
- •Linear Diffusion Equation
- •Nonlinear Models for Image Prcessing
- •Optimization & Work to be Completed

The goal of image processing is to enhance desired features while suppressing undesirable ones

Typically this is accomplished by simply transforming one 2D function into another.

$$u(x,y) \rightarrow u'(x,y)$$

or

$$\vec{u}' = T\left[\vec{u}\right]$$

Usually this is preformed by convolution with a filter function

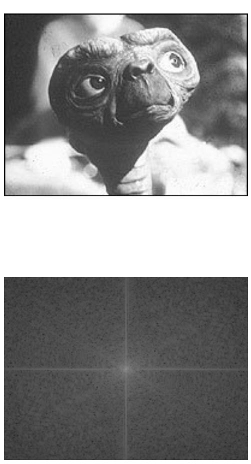
$$u' = \iint u(x,y)h(x-\xi, y-\eta)d\xi d\eta d \equiv u \otimes h$$

Often it is simpler to enter the Fourier domain and filter out undesirable spatial frequencies

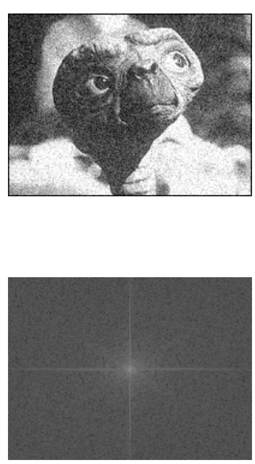
$$FT\{u \otimes h\} = FT\{u\} \cdot FT\{h\}$$

$$u' = FT^{-1}\left\{FT\{u\} \cdot FT\{h\}\right\}$$

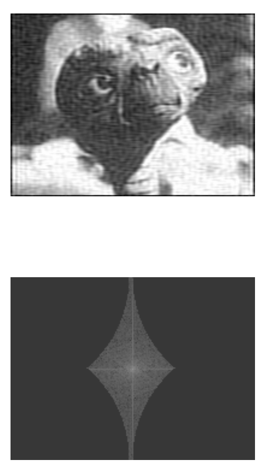
An Example of Fourier Filtering



The Original Image



The Image with Noise



The Filtered Image

Axioms of Image Processing

Supposed we added an artificial time dependence to our image so that we could iteratively transform it.

Architectural axioms:

Causality $$T_{t+s} = T_{t+s,s} \circ T_s \ for\ all\ 0 \le s,t \le \infty$$

Regularity $$T_t\left(f + hg\right) - \left(T_t\left(f\right) + hg\right) \le Cht\ for\ all\ 0 \le s,t \le \infty$$

Locally Smooth

Comparison Principle $$T_t\left(f\right) \le T_t\left(g\right)\ for\ all\ t \ge 0, f \le g$$

Morphological axioms:

Shift Invariance

Grayscale Invariance

$$T_t\left(hf + c\right) = hT_t\left(f\right) + c$$

Axioms and Fundamental Equations of Image Processing
Luis Alvarez, Frederic Guichard, Pierre-Louis Lions & Jean-Michel More
Arch. Rational Mech. Anal. 123 (1993) 199-257. 9 Springer-Verlag 1993

Solutions to the Axioms of Image Processing

$$\frac{\partial u}{\partial t} = F\left[\Delta u, \left|\nabla u\right|, u, t, r\right]$$

The solutions to the axioms are parabolic PDEs

Where: $\Delta u = div\left(\nabla u\right)$

(unless we are doing RGB imaging)

The three terms: u,t,r are often used in physical optical systems for feedback (such as astrophysics telescopes) to remove phase distortions

However, they could lead to instabilities in the image, and often neglected for analysis in most computational systems. In fact all PDE image processors must be checked with a von Neumann like stability analysis

Linear Diffusion

Suppose we want to reduce noise in an image. If we let high intensity pixels diffuse to low intensity pixels we preserve shapes and reduce noise

$$\partial_t u(x,y,t) = \Delta u(x,y,t)$$
$$u(x,y,0) = I(x,y)$$
$$\partial_{x,y} u \,|_{boundary} = 0$$

The solution to the diffusion equation under the specific Neumann boundary condition:

$$u = \iint \frac{1}{4\pi t} e^{\left(-\frac{(x-\xi)^2+(y-\eta)^2}{4t}\right)} I(\xi,\eta) d\xi d\eta, t > 0$$

Yet this is just a convolution filter!

Nonlinear Diffusion

Original image

Perona & Malik came up with a solution to keep the noise reduction properties while detecting edges

$$\frac{\partial u}{\partial t} = div\left(g\left(|\nabla u|\right)\nabla u\right)$$

Image with Noise

$g(x) = 1$

$g(x) = e^{-\frac{x^2}{\sigma^2}}$

$g(x) = \frac{x^2}{1+x^2}$

Nonlinear Diffusion

There have been many models to add nonlinear elements to the diffusion equation:

TABLE I
SUMMARY OF SOME φ FUNCTIONS

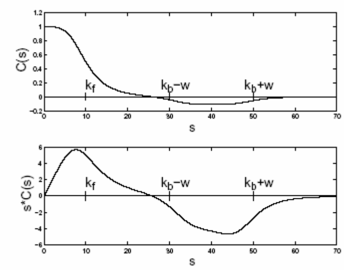
Name of the function φ	$\varphi(t)$	Convexity	$\frac{\varphi'(t)}{2t}$		Conditions (*i-iii*) satisfied
Tikhonov	t^2	yes	1		no
Total Variation [10,11]	t	yes	$\frac{1}{2t}$	(1)	no
Bouman & Sauer [6]	t^α , $1 \leq \alpha \ll 2$	yes	$\frac{\alpha}{2} t^{\alpha-2}$	(1)	no
Perona & Malik [14]	$-\exp(-t^2)+1$	no	$\exp(-t^2)$		yes
Geman & McClure [34]	$\frac{t^2}{1+t^2}$	no	$\frac{1}{(1+t^2)^2}$		yes
Hebert & Leahy [35]	$\log(1+t^2)$	no	$\frac{2}{1+t^2}$		yes
Green [36]	$\log[\cosh(t)]$	yes	$\frac{\tanh(t)}{2t}$		yes
Hyper surfaces [9,25]	$\sqrt{1+t^2}-1$	yes	$\frac{1}{2\sqrt{1+t^2}}$		yes

(1) $\frac{\varphi'(t)}{t}$ undefined if $t=0$

TEBOUL *et al.*: EDGE-PRESERVING REGULARIZATION

$$u_t = -sign\left(G * u_{\eta\eta}\right)\left|\nabla u\right| + cu_{\xi\xi}$$

Osher, Rudin Feature-Oriented Image Enhancement using Shock Filters



Gilboda, Sochen, Zeevi: Forward and Backward Diffusion Processes

Optimal Filter Design

My goal is to use an energy function to optimize the nonlinear diffusion process. However any energy function to create the best image needs to assume you have some knowledge of what the image is suppose to be.

$$J = \iint \left(u_f - u_o\right)^2 dxdy$$

$$\min J$$

$$\partial_t = div\left(g\left(\left|\nabla u\right|, t\right)\nabla u\right)$$

$$u\left(t=0\right) = I$$

$$\partial_\perp u\,|_{boundary} = 0$$

$$g\left(x,t\right) = control$$

Conclusion

•Traditional Methods in Image Processing can be supplemented by PDE image processing

•Mathematical Axioms for Image transformation allow us to develop a framework for good prosperities in PDE design

•Nonlinear Models for Image Prcessing work well for edge enhancement and noise reduction

Thank You!