Exponential stability of switched hyperbolic systems in a bounded domain ⋆

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Abstract

We consider switching in time among a finite family of systems governed by linear hyperbolic partial differential equations on a bounded space interval. The switching system is fairly general in that the space dependent system matrix functions as well as the boundary conditions may switch in time. For the case in which the switching occurs between hyperbolic systems in the canonical diagonal form, we provide two sets of sufficient conditions for the switched system to be exponentially stable under arbitrary switching signals. These results are generalizations of the corresponding results for the un-switched case. Furthermore, we provide an explicit dwell-time bound on the switching signals that guarantee exponential stability of the switched system under the assumption that each of the individual systems are stable. Our results of stability under arbitrary switching generalize to the case in which switching occurs between non-diagonal hyperbolic systems that are diagonalizable using a common transformation.

Key words: Distributed parameter systems; stability of hybrid systems; switched systems.

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1 Introduction

Flows in physical infrastructure networks such as transportation systems [6], irrigation canal systems [17], [4], and gas distribution systems [1] can be modeled by systems of hyperbolic conservation laws in one spatial dimension. These physical networked systems can be monitored by static [19] or mobile sensor networks [7] and controlled at nodes by *supervisory control and data acquisition* (SCADA) systems. A common control problem studied in the context of these conservation laws is the problem of stability and stabilization under boundary control actions. Recent years have witnessed a significant amount of research activity on this topic [9], [8], [25], [17], [4], [3], [2]. From a practical point-of-view, it is of interest to consider situations in which during the period of operation, certain parameters of the system exhibit switching in time triggered by external factors [18]. In addition, a controller based on logical rules may switch between one of the several possible control actions [22]. Such systems involving continuous dynamics and discrete events are known as *hybrid systems* and are extensively studied in the literature when the subsystems are governed by *ordinary differential equations* (ODEs) [18], [15], [24] and *differential algebraic equations* (DAEs) [26] in finite dimensional spaces. Very few attempts have been made to extend this theory to infinite dimensional spaces, with the exception of [23], [21], though hybrid systems in which modes are governed by *partial differential equations* (PDEs) represent a relatively unexplored and potentially rich field of study [10]. While we will here consider switching in time of a certain class of a first order hyperbolic PDE we note that, in general, systems modeled by PDEs may exhibit hybrid behavior in a variety of ways: switching sequentially in time or sequentially in space or distributed in the space/space-time domain. Motivated by above arguments, we focus on stability properties of systems that switch among a finite family of linear hyperbolic PDEs in a bounded domain $[a, b] \subset \mathbb{R}$. We consider n-dimensional vector solutions $u(t, s)$ in the space-time

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strip $\Omega([0,\infty)) := \{(t,s)|t \in [0,\infty), s \in [a,b]\}$ with dynamics:

$$\begin{cases} \dfrac{\partial u}{\partial t} + A^j(s)\dfrac{\partial u}{\partial s} + B^j(s)u = 0, \ s \in (a,b), \ t > 0 \\ D_L^j u(t,a) = 0, \quad D_R^j u(t,b) = 0, \ t \in [0,\infty) \\ u(0,s) = \bar{u}(s), \quad s \in (a,b) \end{cases} \tag{1}$$

where j belongs to a discrete set $\mathcal{Q} = \{1,\ldots,N\}$ and

- $A^j(s)$ and $B^j(s)$ are space dependent matrix functions in $\mathbb{R}^{n\times n}$ for mode j,
- D_L^j and D_R^j are matrices of appropriate size specifying data at the boundary of $[a,b]$ in mode j, and $\bar{u}(s)$ specifies the initial data on (a,b).

Switching occurs in time according to a piecewise constant switching signal $\sigma(\cdot)$: $\mathbb{R}^+ \to \mathcal{Q}$ with switching times $\tau_k \in \mathbb{R}^+$ ($k \in \mathbb{N}$) at which $\sigma(\cdot)$ switches discontinuously from one mode $j \in \mathcal{Q}$ to another mode $j' \in \mathcal{Q}$. We study two specific problems:

(A) Find conditions that guarantee exponential stability of the switched PDE for arbitrary switching signals.
(B) Alternatively, characterize a (preferably large) class of switching signals such that the switched PDE is exponentially stable.

These problems are relevant when the switching mechanism is either unknown or too complicated for a more careful stability analysis, in particular when the switching happens autonomously as for instance in networked transport systems [5].

Without switching, the system (1) is well understood [13], [14], e. g. it is known that the associated linear operators generate C_0-semigroups. Exponential stability without switching has been considered in [8] and, more general for quasi-linear hyperbolic systems, in [25]. Furthermore, a sufficient condition for switching infinitesimal generators of semigroups is given in [23] and the problem of stability of discontinuous dynamical systems determined by semigroups was studied in [21]. However, we justify our addition, noting that to the best knowledge of the authors, these known results cannot be applied to the specific equations considered here, lacking a necessary commutativity assumption of the infinitesimal generators due to the presence of (changing) boundary conditions and a sufficiently tight growth bound for the semigroups generated.

The article is organized as follows. In Section 2, we discuss the well-posedness of switching among the systems in (1) and discuss preliminaries for stability analysis of the switched system. In view of Problem (A), Section 3.1 generalizes existing results of [8] and [25] for the exponential stability of hyperbolic systems to the case when switching occurs between systems that are in the canonical diagonal form. In view of problem (B), we provide a dwell-time bound such that the system is exponentially stable for slow enough switching. Of particular interest is the sufficient condition for exponential stability in terms of a set of spectral radius conditions involving boundary data. In Section 3.2, we show that this sufficient condition is no longer sufficient for exponential stability of systems when switching occurs between non-diagonal systems. We also show that the results for switching between diagonal systems extend to the case of non-diagonal systems when all systems are simultaneously diagonalizable using a common transformation. We discuss a potential application to stability of water flow in one-dimensional open channels governed by linearized Saint-Venant equations in Section 4. Some final remarks are drawn in Section 5.

2 Preliminaries

2.1 Well-posedness of the switched hyperbolic system

We will be considering switching among the systems (1) in the Hilbert space $\mathcal{H} = L^2((a,b),\mathbb{R}^n)$ (also denoted as $(L^2(a,b))^n := L((a,b),\mathbb{R}) \times \ldots \times L((a,b),\mathbb{R})$). We first recall useful results for the unswitched case, see [9], [12],[13], [14] for details. It is known that for a fixed mode $j \in \mathcal{Q}$, the corresponding system in (1) is well-posed under the following assumptions:

(H_1) The matrix functions $A^j(s)$ and $B^j(s)$ have the property that $A^j(\cdot) \in C^1([a,b],\mathbb{R}^{n\times n})$, $B^j(\cdot) \in C^0([a,b],\mathbb{R}^{n\times n})$, and for all $s \in [a,b]$, there exists m_j such that $0 < m_j < n$ and $A^j(s)$ has m_j negative and $(n-m_j)$ positive eigenvalues $\lambda_i^j(s)$ with n corresponding linearly independent left (resp. right) eigenvectors $l_i^j(s)$ (resp. $r_i^j(s)$), $i = 1,\ldots,n$. Here $\lambda_i^j(\cdot) \in C^1([a,b],\mathbb{R})$ and $l_i^j(\cdot)$, $r_i^j(\cdot) \in C^1([a,b],\mathbb{R}^n)$.
(H_2) The following rank conditions hold for $D_L^j \in \mathbb{R}^{(n-m_j)\times n}$ and $D_R^j \in \mathbb{R}^{m_j\times n}$

$$\begin{aligned} \operatorname{rank}\big[(D_L^j)^\top \big| l_1^j(a)\big| \cdots \big| l_{m_j}^j(a)\big] &= n \\ \operatorname{rank}\big[(D_R^j)^\top \big| l_{m_j+1}^j(b)\big| \cdots \big| l_n^j(b)\big] &= n. \end{aligned} \tag{2}$$

That is, for a fixed mode $j \in \mathcal{Q}$ and initial condition $\bar{\mathbf{u}} := \bar{u}(\cdot) \in \mathcal{H}$, under the hypotheses ($\mathrm{H}_1$) and ($\mathrm{H}_2$), a solution $\mathbf{u}^j(\cdot) = u^j(\cdot,\cdot) \in C^0([0,\infty),\mathcal{H})$ of the corresponding system in (1) exists and is unique (see e.g. ([12], Sections 2.1 and 2.3), [9]). This solution is to be interpreted in the weak sense. Furthermore, the matrix functions $S_j(\cdot) = [l_1^j(\cdot)\big| \ldots \big| l_n^j(\cdot)]^\top$ and $S_j^{-1}(\cdot) = [r_1^j(\cdot)\big| \ldots \big| r_n^j(\cdot)]^\top$ in $C^1([a,b],\mathbb{R}^{n\times n})$ are such that for all $s \in [a,b]$

$$S_j(s)A^j(s)S_j^{-1}(s) = \Lambda^j(s), \tag{3}$$

where $\Lambda^j(s) = \mathrm{diag}(\Lambda_I^j(s), \Lambda_{II}^j(s)) \in \mathbb{R}^{n\times n}$ with

$$\begin{aligned}\Lambda_I^j(s) &= \mathrm{diag}(\lambda_1^j(s),\dots,\lambda_{m_j}^j(s)) < 0\\ \Lambda_{II}^j(s) &= \mathrm{diag}(\lambda_{m_j+1}^j(s),\dots,\lambda_n^j(s)) > 0.\end{aligned}$$

For a fixed mode $j \in \mathcal{Q}$, we can obtain a convenient *diagonal form* for the corresponding system in (1). By applying the following transformation

$$\begin{aligned}u^j(t,s) &= S_j^{-1}(s)\xi^j(t,s)\\ \tilde{D}_L^j &= D_L^j S_j^{-1}(a)\\ \tilde{D}_R^j &= D_R^j S_j^{-1}(b).\end{aligned}$$

and using the representation

$$\begin{aligned}&\xi^j(t,s) = \begin{pmatrix}\xi_I^j(t,s)\\ \xi_{II}^j(t,s)\end{pmatrix},\\ &\tilde{B}^j(s) = S_j(s)\left(A^j(s)\frac{\partial}{\partial s}S_j^{-1}(s) + B^j(s)S_j^{-1}(s)\right),\\ &\tilde{D}_L^j = [\tilde{D}_{L,I}^j | \tilde{D}_{L,II}^j], \quad \tilde{D}_R^j = [\tilde{D}_{R,I}^j | \tilde{D}_{R,II}^j],\\ &G_L^j = -(\tilde{D}_{L,II}^j)^{-1}\tilde{D}_{L,I}^j, \quad G_R^j = -(\tilde{D}_{R,I}^j)^{-1}\tilde{D}_{R,II}^j, \quad (4)\end{aligned}$$

where $\xi_I^j(t,\cdot) \in L^2((a,b),\mathbb{R}^{m_j}), \xi_{II}^j(t,\cdot) \in L^2((a,b),\mathbb{R}^{n-m_j})$ and

$$\begin{aligned}&\tilde{D}_{L,I}^j \in \mathbb{R}^{(n-m_j)\times m_j}, \tilde{D}_{L,II}^j \in \mathbb{R}^{(n-m_j)\times(n-m_j)},\\ &\tilde{D}_{R,I}^j \in \mathbb{R}^{m_j\times m_j}, \tilde{D}_{R,II}^j \in \mathbb{R}^{m_j\times(n-m_j)},\\ &G_L^j \in \mathbb{R}^{(n-m_j)\times m_j}, G_R^j \in \mathbb{R}^{m_j\times(n-m_j)},\end{aligned}$$

the system corresponding to mode j in (1) becomes

$$\begin{cases}\dfrac{\partial \xi^j}{\partial t} + \Lambda^j(s)\dfrac{\partial \xi^j}{\partial s} + \tilde{B}^j(s)\xi^j = 0\\ \xi_{II}^j(t,a) = G_L^j\xi_I^j(t,a), \quad \xi_I^j(t,b) = G_R^j\xi_{II}^j(t,b)\\ \xi^j(0,s) = \bar{\xi}^j(s) = S_j(s)\bar{u}(s).\end{cases} \tag{5}$$

The wellposedness of the diagonal form (5) for the system corresponding to mode j can be argued by observing that each component ξ_i^j, $i = 1,\dots,n$ of ξ^j satisfies a differential equation along the C^1-curve, called *characteristic curve*, defined by the characteristic equation

$$\frac{ds_i^j(t)}{dt} = \lambda_i^j(s_i^j(t)). \tag{6}$$

Along each curve $s_i^j(t)$, we have

$$\frac{d}{dt}\xi_i^j(t,s_i^j(t)) = -\sum_{k=1}^{n} b_{ik}^j(s_i^j(t))\xi_k^j(t,s_i^j(t)) \tag{7}$$

where b_{ik}^j is the element corresponding to ith row and kth column of $\tilde{B}^j$. Following [9], we note that for a given initial condition $\bar{\xi}^j(\cdot) \in \mathcal{H}$, the solution of these equations is a unique $C^0([0,\infty),\mathcal{H})$ solution of (5), denoted by $\boldsymbol{\xi}^j(t) = (\xi_I^j(t,\cdot)^\top, \xi_{II}^j(t,\cdot)^\top)^\top$.
We will now argue the well-posedness of the system (1) under a given switching signal $\sigma(\cdot)$. We assume non-zenoness for the switching signals, i. e.,

(H$_3$) All switching signals $\sigma(\cdot)$ are such that there are only finitely many switches $j \curvearrowright j'$ in each finite time interval of $\mathbb{R}^+ = \{t \geq 0\}$.

The above assumption is commonly made in the field of switched and hybrid systems to avoid zeno behavior that is anticipated with the accumulation of switching times leaving unclear the continuation of the continuous dynamics beyond such accumulation points. Any given (non-zeno) switching signal $\sigma(\cdot)$ defines a mode $j_k \in \mathcal{Q}$ for each interval $[\tau_k, \tau_{k+1})$. For an initial condition, $\bar{\mathbf{u}} = \bar{u}(\cdot) \in \mathcal{H}$, we define $\mathbf{u}(t) = u(t,\cdot)$ as

$$u(t,\cdot) = u^{j_k}(t,\cdot), \quad \text{for } t \in (\tau_k, \tau_{k+1}) \text{ a.e. on } (a,b) \tag{8}$$

where $u^{j_k}(t,\cdot)$ is a solution of the system corresponding to mode j_k in (1) with the initial condition

$$u^{j_k}(\tau_k,\cdot) = \begin{cases}\lim_{t\to\tau_k, t<\tau_k} u^{j_{k-1}}(t,\cdot) & \text{if } k > 0,\\ \bar{u}(\cdot) \quad \text{if } k = 0.\end{cases} \tag{9}$$

Under hypotheses (H$_1$), (H$_2$), and additionally (H$_3$) for any switching signal $\sigma(\cdot)$, the solution $\mathbf{u}(t)$ as constructed in (8) and (9) exists and we have $\mathbf{u}(\cdot) \in C^0([0,\infty),\mathcal{H})$. For each mode $j \in \mathcal{Q}$, uniqueness follows from [12], [9]. Uniqueness of the solution of (1) results from the construction at switching times τ_k following (9).
Alternatively, we also present a representation of the solution $\mathbf{u}(t)$ as a *discontinuous dynamical system* (DDS) determined by semigroups [21]. For a fixed mode $j \in \mathcal{Q}$, we define the following (unbounded) operator $\mathcal{A}^j\colon \mathcal{H} \supset D^j(\mathcal{A}^j) \longrightarrow \mathcal{H}$ associated with system (5) by

$$\begin{cases}\mathcal{A}^j\begin{pmatrix}f_I(s)\\ f_{II}(s)\end{pmatrix} := -\Lambda^j(s)\dfrac{\partial}{\partial s}\begin{pmatrix}f_I(s)\\ f_{II}(s)\end{pmatrix} - \tilde{B}^j(s)\begin{pmatrix}f_I(s)\\ f_{II}(s)\end{pmatrix},\\ D^j(\mathcal{A}^j) := \Big\{f = (f_I^\top, f_{II}^\top)^\top \in (\mathcal{H}^1(a,b))^n \mid\\ f_{II}(a) = G_L^j f_I(a),\ f_I(b) = G_R^j f_{II}(b)\Big\}.\end{cases} \tag{10}$$

With (10), for any mode $j \in \mathcal{Q}$, (5) can be written as an evolution equation on $\mathcal{H}$:

$$\frac{d\boldsymbol{\xi}^j(t)}{dt} = \mathcal{A}^j\boldsymbol{\xi}^j(t), \quad t > 0. \tag{11}$$

It is well known from [20] that for a fixed $j \in \mathcal{Q}$, the operator (10) generates a C_0-semigroup $\{T^j(t)\}_{t\geq 0}$ on $\mathcal{H}$.

Thus, for a given initial condition $\bar{\mathbf{u}} := \bar{u}(\cdot) \in \mathcal{H}$ and any switching signal $\sigma(\cdot)$ that defines a mode j_k for each interval $[\tau_k, \tau_{k+1})$, the solution $\mathbf{u}$ of switching among the original systems (1), as defined by (8) and (9), can be represented as a DDS determined by semigroups

$$\begin{aligned} \mathbf{u}(t) &= S_{j_k}^{-1}(\cdot)T^{j_k}(t-\tau_k)S_{j_k}(\cdot)\mathbf{u}(\tau_k), \quad \tau_k < t < \tau_{k+1} \\ \mathbf{u}(\tau_k) &= \lim_{t\to\tau_k, t<\tau_k} \mathbf{u}(t), \quad k = 1, 2, \ldots \end{aligned} \tag{12}$$

starting with $\mathbf{u}(\tau_0, \cdot) = \mathbf{u}(0) = \bar{u}(\cdot)$ and $\tau_0 = 0$. System (12) has been obtained from the solution of (11), which can be written in the ξ coordinates for the initial condition $\boldsymbol{\xi}(0) = \bar{\xi}(\cdot) := S_0(\cdot)\bar{u}(\cdot)$ and the switching signal $\sigma(\cdot)$ as the DDS

$$\begin{aligned} \boldsymbol{\xi}(t) &= T^{j_k}(t-\tau_k)\boldsymbol{\xi}(\tau_k), \quad \tau_k < t < \tau_{k+1} \\ \boldsymbol{\xi}(\tau_k) &= S_{j_{k+1}}(\cdot)S_{j_k}^{-1}(\cdot) \lim_{t\to\tau_k, t<\tau_k} \boldsymbol{\xi}(t), k = 1, 2, \ldots \end{aligned} \tag{13}$$

Remark 1 *Note that because of the well-posedness the switched system* (1) *under any switching signal* $\sigma(\cdot)$, *for every* $T > 0$, *there exists a constant* $C(T) > 0$ *such that*

$$\|\mathbf{u}(t)\|_{L^2((a,b),\mathbb{R}^n)} \leq C(T)\|\mathbf{u}(0)\|_{L^2((a,b),\mathbb{R}^n)}, \forall t \in [0, T]. \tag{14}$$

See for example [11], [12], Section 2.1 *and* 2.3 *for the case of a single mode* j. *The extension to the switched case is straightforward.*

While proving main results in Section 3.1, we will use the following hypothesis which is a special case of H_1:

(H_1^*) In addition to H_1, for all $j \in \mathcal{Q}$ and $s \in [a, b]$, $A^j(s)$ has exactly m negative and $(n-m)$ positive eigenvalues, independent of j, where $0 < m < n$.

2.2 Definitions and notation

For the specific initial condition $\bar{u} \equiv 0$ and any switching signal $\sigma(\cdot)$, the solution $u(t, \cdot)$ of switching among the systems (1) satisfies $u(t, \cdot) = 0$ for all $t \geq 0$. Without loss of generality, we will be considering this as the only *equilibrium state*. For a given switching sequence $\sigma(\cdot)$, we say that the system is *exponentially stable* with respect to a norm $\|\cdot\|$ if there exists $c > 0$ and $\beta > 0$ such that for every initial condition $u(0, \cdot)$, the solution $u(t, \cdot)$ of the switched system (1) satisfies

$$\|u(t, \cdot)\| \leq c \exp(-\beta t)\|u(0, \cdot)\|. \tag{15}$$

In view of main problem (A), we then say that the switched system is *absolutely exponentially stable* if it is exponentially stable for all switching sequences $\sigma(\cdot)$ satisfying assumption (H_3).

In view of problem (B), we say that a value $\tau > 0$ is a *dwell-time* of a switching signal $\sigma(\cdot)$, if the intervals between consecutive switches are no shorter than τ, that is, $\tau_{k+1} - \tau_k \geq \tau$ for all $k > 0$.

We also define the following values for system (5)

$$\begin{aligned} \bar{\tau}_j :=&(b-a)\Bigg\{\left(\min_{i=1,\ldots,m_j} \min_{s\in[a,b]} |\lambda_i^j(s)|\right)^{-1} \\ &+ \left(\min_{i=m_j+1,\ldots,n} \min_{s\in[a,b]} |\lambda_i^j(s)|\right)^{-1}\Bigg\} \\ \bar{\tau} :=& \max_{j\in\mathcal{Q}} \bar{\tau}_j. \end{aligned} \tag{16}$$

Intuitively, $\bar{\tau}$ is an upper bound of the time in which the slowest of all possible characteristic curves will have undergone reflections at both boundaries.

For a matrix $M \in \mathbb{R}^{n\times n}$, $|M|$ is used to denote the matrix whose elements are absolute values of elements of M, i. e., $|M| = (|M_{ij}|)$. The infinity norm and the infinity norm associated with diagonal scaling are respectively defined as

$$\|M\|_\infty := \max\{\sum_{j=1}^{n} |M_{ij}|; i \in \{1, \ldots, n\}\}$$
$$\Theta(M) := \inf\{\|\gamma M \gamma^{-1}\|_\infty; \gamma = diag\{\gamma_i\}, \gamma_i > 0\}$$

and it is known from ([25], Lemma 2.4, page 146) that:

$$\Theta(M) = \rho(|M|)$$

where $\rho(|M|)$ is the spectral radius of $|M|$. In the context of the switching system (1), we define N^2 *characterizing matrices* as follows: for each pair $(j, j') \in \mathcal{Q}\times\mathcal{Q}$ we set

$$G^{jj'} := \begin{pmatrix} 0 & G_R^{j'} \\ G_L^j & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}. \tag{17}$$

3 Stability of switched hyperbolic system

3.1 Stability of switched hyperbolic system in diagonal form

We first consider the simpler problem of switching among systems which are in diagonal form, i. e., we make the following assumption:

Assumption 1 *We assume that all* $A^j(\cdot)$ *are diagonal matrix functions, i. e.,*

$$A^j(\cdot) = \Lambda^j(\cdot) = \mathrm{diag}(\lambda_1^j(\cdot), \ldots, \lambda_n^j(\cdot)).$$

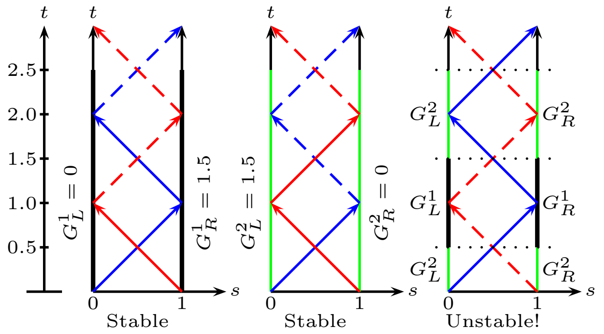

Fig. 1. Instability by switching.

Remark 1 *Note that Assumption 1 and* (H_1) *imply that* $S_j = S_j^{-1} = \mathbb{I}$ *(identity) and that the switching system* (1) *with* $j = \sigma(\cdot)$ *is equivalent to* (5) *with* $j = \sigma(\cdot)$ *having* $\Lambda^j = A^j$ *and* $\tilde{B^j} = B^j$*. For this case, we define* $u_I(t,\cdot) := \xi_I(t,\cdot)$, $u_{II}(t,\cdot) := \xi_{II}(t,\cdot)$, $A_I(\cdot) := \Lambda_I(\cdot)$, *and* $A_{II}(\cdot) := \Lambda_{II}(\cdot)$*.*

We denote $s_i^{\sigma(\cdot)}(\cdot)$ as the characteristic path defined by the concatenation of the characteristic curves $s_i^j(\cdot)$ throught the switching times. Also, observe that if in addition (H_1^*) holds, the characteristic paths can be classified into left- and right-going, independently of the switching signal $\sigma(\cdot)$.

The results we present in this section are motivated by a simple PDE counterpart to the classical ODE observation [18] that exponential stability of all subsystems is not sufficient for the exponential stability of the switched system. This observation holds even under the Assumption 1 as illustrated by the following example:

Example 2 *Let* $\mathcal{Q} = \{1, 2\}$, $[a, b] = [0, 1]$,

$$\Lambda^j = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix},\ G_L^j = 1.5(j-1),\ G_R^j = 1.5(2-j),$$

and consider $\bar{\boldsymbol{\xi}}(s) = 1$ *for* $s \in (a, b)$*. For the case of no switching, the solution of the system* $\boldsymbol{\xi}(\cdot)$ *for* $j = 1$ *and* $j = 2$ *is zero for all* $t > 2$*, but the solution of the system with a switching signal* $\sigma(t)$ *that is defined over the switching times*

$$\tau_k = 0.5, 1.5, 2.5, \ldots \tag{18}$$

and alternates between modes in $\mathcal{Q}$ *starting with* $\sigma(0) = 2$ *is not stable, i. e.* $\|\boldsymbol{\xi}(t)\|_{L^\infty((a,b),\mathbb{R}^2)}$ *is not bounded as* $t \to \infty$*, because the values on the right-going characteristic emerging from* $s \in (0, 0.5)$ *always increase by reflection of the characteristics along the boundary. See Figure 1.* □

It was shown in [23], that infinite dimensional switched systems such as (11) are exponentially stable for arbitrary switching if all subsystems are exponentially stable and the operators $\mathcal{A}^j$ commute pairwise. However, due to the presence of switching boundary conditions in (1), the operators $\mathcal{A}^j$ defined in (10) do not commute pairwise in general. Thus we will focus on conditions for the boundary data under which the switched system is absolutely exponentially stable. We begin with a very strong sufficient condition where we require all boundary data to be *strictly dissipative* [8].

Theorem 3 *For any initial condition* $\bar{u} \in \mathcal{H}$*, consider switching among the systems* (1) *under the Assumption 1. If under hypotheses* (H_1^*)*,* (H_2) *and* (H_3)*, the solution* $\mathbf{u}(t) = (u_I(t,\cdot)^\top, u_{II}(t,\cdot)^\top)^\top$ *under any mode* j *satisfies*

$$\begin{pmatrix} u_I(t,s) \\ u_{II}(t,s) \end{pmatrix}^\top \left(\frac{\partial A^j(s)}{\partial s} - B^j(s) - B^j(s)^\top \right) \begin{pmatrix} u_I(t,s) \\ u_{II}(t,s) \end{pmatrix} \le 0 \text{ for all } s \in (a,b) \tag{19}$$

$$u_I^\top(t,a)(A_I^j(a) + (G_L^j)^\top A_{II}^j(a) G_L^j) u_I(t,a) \le -r_I \|u_I(t,a)\|^2_{\mathbb{R}^m} \tag{20}$$

$$u_{II}(t,b)^\top (A_{II}^j(b) + (G_R^j)^\top A_I^j(b) G_R^j) u_{II}(t,b) \ge r_{II} \|u_{II}(t,b)\|^2_{\mathbb{R}^{n-m}} \tag{21}$$

for all $j \in Q$*, where the matrices* G_L^j *and* G_R^j *are computed according to* (4)*,* A_I^j*,* A_{II}^j *are defined in Remark 1, and the constants* $r_I, r_{II} \ge 0$ *are such that* $r_I + r_{II} > 0$*, then the switched system is absolutely exponentially stable with respect to the* L^2*-norm, i.e., there exists* $\beta > 0$ *and* $c > 0$ *such that for every initial condition* $\bar{u} \in \mathcal{H}$*, the solution* $\mathbf{u}(\cdot)$ *of* (1) *satisfies*

$$\|\mathbf{u}(t)\|_{L^2((a,b),\mathbb{R}^n)} \le c \exp(-\beta t) \|\bar{u}\|_{L^2((a,b),\mathbb{R}^n)}.$$

Proof Under Assumption 1 and referring to Remark 1, we have $\mathbf{u}(\cdot) = \boldsymbol{\xi}(\cdot)$. We suppose $r_I > 0$. The case when $r_{II} > 0$ is analogous. We have

$$\begin{aligned}
\frac{d}{dt}\|\boldsymbol{\xi}(t)\|^2_{L^2((a,b),\mathbb{R}^n)} &= \frac{d}{dt}\int_a^b \xi^\top(t,s)\xi(t,s)ds \\
&= -2\int_a^b \xi^\top(t,s)\Lambda^j(s)\frac{\partial}{\partial s}\xi(t,s)ds \\
&\quad - \int_a^b \xi^\top(t,s)\left(\tilde{B}^j(s) + \tilde{B}^j(s)^\top\right)\xi(t,s)ds \\
&= -\xi(t,b)^\top\Lambda^j(b)\xi(t,b) + \xi(t,a)^\top\Lambda^j(a)\xi(t,a) \\
&\quad + \int_a^b \xi(t,s)^\top\left(\frac{\partial\Lambda^j(s)}{\partial s} - \tilde{B}^j(s) - \tilde{B}^j(s)^\top\right)\xi(t,s)ds \\
&\le -\xi(t,b)^\top\Lambda^j(b)\xi(t,b) + \xi(t,a)^\top\Lambda^j(a)\xi(t,a)
\end{aligned}$$

where we replaced $\frac{d}{dt}\xi(t,s)$ and $\frac{d}{dt}\xi(t,s)^\top$ using (5) and the inequality follows from condition (19). Further, using the boundary conditions in (5) we get

$$\begin{aligned}&\frac{d}{dt}\|\boldsymbol{\xi}(t)\|^2_{L^2((a,b),\mathbb{R}^n)}\\&\le -\xi_{II}^\top(t,b)\left(\Lambda^j_{II}(b)+(G^j_R)^\top\Lambda^j_I(b)G^j_R\right)\xi_{II}(t,b)\\&\quad+\xi_I^\top(t,a)\left(\Lambda^j_I(a)+(G^j_L)^\top\Lambda^j_{II}(a)G^j_L\right)\xi_I(t,a)\\&\le -r_I\|\xi_I(t,a)\|^2\end{aligned}$$

where the inequality follows from conditions (20) and (21). Thus, we have

$$\|\boldsymbol{\xi}(t)\|^2_{L^2((a,b),\mathbb{R}^n)}\le\|\boldsymbol{\xi}(0)\|^2_{L^2((a,b),\mathbb{R}^n)}-r_I\int_0^t\|\xi_I(\vartheta,a)\|^2d\vartheta. \tag{22}$$

So we see that $\|\boldsymbol{\xi}(t)\|_{L^2((a,b),\mathbb{R}^n)}$ is a non-increasing function of t for all switching signals $\sigma(\cdot)$. Consider $t^*>\bar\tau$ (say $t^*=\bar\tau+p$) where $\bar\tau$ is given by (16) and p is a small positive number. Fix any $j\in\mathcal{Q}$ and let t^j_u be the time at which the slowest characteristic path ending up at (a,t^*) starts from (b,t^j_u) if the system was in mode j all the time and set $t_u=\min_{j\in\mathcal{Q}}t^j_u$. Similarly, let t^j_l be the time at which the slowest characteristic starting from $(a,0)$ ends up at (b,t^j_l) if the system was in mode j all the time and set $t_l=\max_{j\in\mathcal{Q}}t^j_l$. By the choice of $t^*>\bar\tau$ we have $t_l<t_u$. See Figure 2 for an illustration of t^*,t_u,t_l and other variables used in the proof.

Similar to Theorem 3, page 86 in [8], we use Theorem 4.3, page 41 in [16] to obtain an energy estimate for the equations $\partial_s\xi=-(\Lambda^j(s))^{-1}\partial_t\xi-(\Lambda^j(s))^{-1}\tilde B^j(s)\xi$ where, by treating the direction of increasing space as time-like, we obtain

$$\int_{t_l}^{t_u}\|\boldsymbol{\xi}(y)\|^2_{L^2((a,b),\mathbb{R}^n)}dy\le\max_{j\in\mathcal{Q}}\{K^j\}\int_0^{t^*}\|\xi_I(\vartheta,a)\|^2d\vartheta,$$

where $K^j>0$ are constants corresponding to the previous inequality for each individual mode, as in [8]. Using that $\|\boldsymbol{\xi}(\cdot)\|_{L^2((a,b),\mathbb{R}^n)}$ is non-increasing independently of $\sigma(\cdot)$ and $t^*>\bar\tau$, we have as in [8] that

$$(t_u-t_l)\|\boldsymbol{\xi}(t^*)\|^2_{L^2((a,b),\mathbb{R}^n)}\le\int_{t_l}^{t_u}\|\boldsymbol{\xi}(y)\|^2_{L^2((a,b),\mathbb{R}^n)}dy.$$

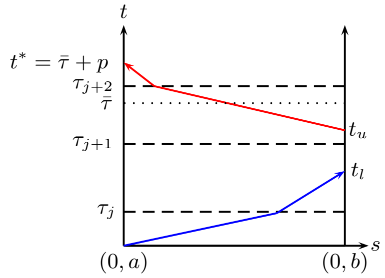

Fig. 2. Choosing t^*, t_u, t_l.

Our final statement can now be established from the formula above by induction. The above estimates are on the time interval $[0,t^*]$ and inequality (22) is on $[0,t]$ for all t. Writing similar estimates on $[t^*,t^*+\bar\tau]$, and using the constant $\gamma=\frac{r_I(t_u-t_l)}{(\max_{j\in\mathcal{Q}}K^j)}>0$ not depending on $\sigma(\cdot)$, we have

$$\begin{aligned}&\|\boldsymbol{\xi}(t^*+\bar\tau)\|^2_{L^2((a,b),\mathbb{R}^n)}\\&\le\|\boldsymbol{\xi}(t^*)\|^2_{L^2((a,b),\mathbb{R}^n)}-\gamma\|\boldsymbol{\xi}(t^*+\bar\tau)\|^2_{L^2((a,b),\mathbb{R}^n)}.\end{aligned}$$

With the constant $\kappa=\sqrt{1/(1+\gamma)}<1$, this implies

$$\|\boldsymbol{\xi}(t^*+\bar\tau)\|_{L^2((a,b),\mathbb{R}^n)}\le\kappa\|\boldsymbol{\xi}(t^*)\|_{L^2((a,b),\mathbb{R}^n)}$$

for all switching signals $\sigma(\cdot)$ satisfying (H_3). Thus, by induction and using (14), we have

$$\begin{aligned}\|\boldsymbol{\xi}(t^*+i\bar\tau)\|_{L^2((a,b),\mathbb{R}^n)}&\le\kappa^i\|\boldsymbol{\xi}(t^*)\|_{L^2((a,b),\mathbb{R}^n)}\\&\le C(t^*)\kappa^i\|\boldsymbol{\xi}(0)\|_{L^2((a,b),\mathbb{R}^n)}\\&=C(t^*)\exp(i\ln(\kappa))\|\boldsymbol{\xi}(0)\|_{L^2((a,b),\mathbb{R}^n)}\end{aligned}$$

and finally, for a suitable constant $c>C(t^*)>0$,

$$\|\boldsymbol{\xi}(t)\|_{L^2((a,b),\mathbb{R}^n)}\le c\exp(-t|\ln(\kappa)|)\|\boldsymbol{\xi}(0)\|_{L^2((a,b),\mathbb{R}^n)}.$$

□

Remark 4 *Theorem 3 also holds if the assumption* $r_I+r_{II}>0$ *is dropped but the inequality* (19) *for all* $j\in\mathcal{Q}$ *is strict for some* $s\in(a,b)$.

In the following we prove another sufficient condition for absolute exponential stability.

Theorem 5 *For any initial condition* $\bar u\in\mathcal{H}$*, consider switching among the systems* (1) *under the Assumption 1. If under hypotheses* (H_1^*)*,* (H_2) *and* (H_3)*, the following condition holds for all* $j,j'\in\mathcal{Q}$

$$\rho(|G^{jj'}|)<1 \tag{23}$$

then there exists $\epsilon>0$ *such that if* $\|B^j(s)\|_\infty<\epsilon$ *for all* $s\in[a,b]$ *and* $j\in\mathcal{Q}$*, then the switched system is*

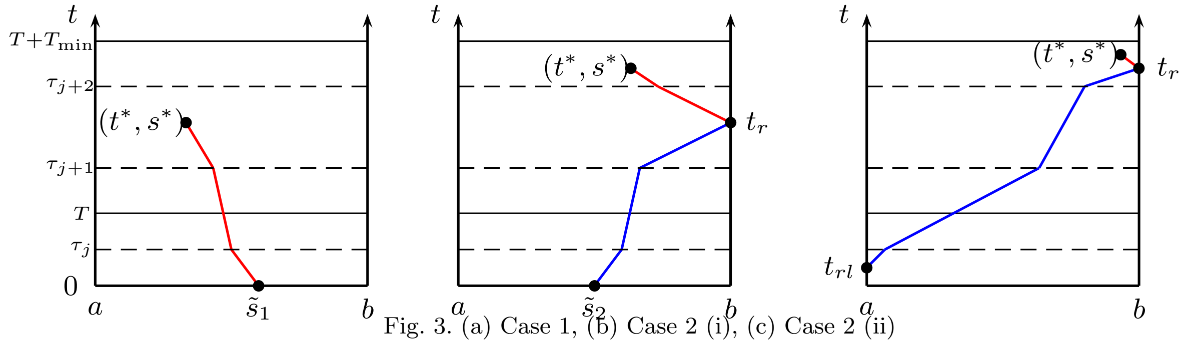

Fig. 3. (a) Case 1, (b) Case 2 (i), (c) Case 2 (ii)

absolutely exponentially stable with respect to the L^∞-norm.

Proof We adapt the arguments of Li [25] Lemma 2.1 and Lemma 2.3 for C^0 solutions of quasilinear systems to the present setting. Again, under the Assumption 1, $\mathbf{u}(\cdot) = \boldsymbol{\xi}(\cdot)$. Following Lemma 2.1 of Li [25] and noting (17), the condition (23) implies

$$\begin{aligned}\theta &:= \max_{j,j'\in\mathcal{Q}} \{|||G_L^j||G_R^{j'}|||_\infty, |||G_R^{j'}||G_L^j|||_\infty\} \\ &= \max_{\substack{r=1,\dots,m\\ l=m+1,\dots,n\\ j,j'\in\mathcal{Q}}} \left\{ \sum_{p=1}^{m}\sum_{k=m+1}^{n} |g_{rk}^{R,j'}||g_{kp}^{L,j}|, \sum_{k=m+1}^{n}\sum_{p=1}^{m} |g_{lp}^{L,j}||g_{pk}^{R,j'}| \right\} \\ &< 1,\end{aligned} \tag{24}$$

where $G_L^j = (g_{pq}^{L,j})$ and $G_R^{j'} = (g_{pq}^{R,j'})$. It suffices to show that there exists $\beta > 0$ and $c > 0$ such that for any initial condition $\bar{\xi} \in \mathcal{H}$, if (24) holds, then there exists $\epsilon > 0$ such that if $\|B^j(s)\| \leq \epsilon$ for all $j \in \mathcal{Q}$, then

$$\|\boldsymbol{\xi}(t)\|_{L^\infty((a,b),\mathbb{R}^n)} \leq c\exp(-\beta t)\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)} \tag{25}$$

where $\boldsymbol{\xi}(\cdot)$ solves of (1) for any given switching signal $\sigma(\cdot)$ (specialized to diagonal form under Assumption 1). Let

$$T_{\min} := (b-a)\Big(\max_{\substack{a\leq s\leq b\\ i=1,\dots,n\\ j=1,\dots,N}} |\lambda_i^j(s)|\Big)^{-1}$$

$$T_{\max} := (b-a)\Big(\min_{\substack{a\leq s\leq b\\ i=1,\dots,n\\ j=1,\dots,N}} |\lambda_i^j(s)|\Big)^{-1}$$

We define the space-time strip $\Omega([t_1,t_2)) := \{(t,s) | t \in [t_1,t_2), s \in [a,b]\}$. To show (25), it suffices to prove that for any fixed $T > 0$, if (25) holds on $\Omega([0,T])$, then it still holds on domain $\Omega([0,T+T_{\min}])$. We define

$$\widehat{\|\boldsymbol{\xi}(t)\|}_{L^\infty((a,b),\mathbb{R}^n)} := \exp(\beta t)\|\boldsymbol{\xi}(t)\|_{L^\infty((a,b),\mathbb{R}^n)}. \tag{26}$$

Assume that (25) holds on $\Omega([0,T])$ and fix some $(t^*,s^*) \in \Omega([T,T+T_{\min}])$. For any $\sigma(\cdot)$, let $z_i(t;t^*,s^*)$ denote the ith characteristic path passing through point $(t^*,s^*) \in \Omega([0,\infty))$, $(i = 1,\dots,n)$ [1]. Then, we have

$$\frac{dz_i(t;t^*,s^*)}{dt} = \lambda_i^{\sigma(t)}(z_i(t;t^*,s^*))$$
$$z_i(t^*;t^*,s^*) = s^*.$$

For any fixed $r = 1,\dots,m$, consider the r-th characteristic path $z_r(t;t^*,s^*)$ passing through (t^*,s^*). Backwards in time, $z_r(t;t^*,s^*)$ either intersects $t = 0$ within the interval $[a,b]$ before hitting any boundary (case 1) or it intersects the line $s = b$ (case 2). See Figure 3 for an illustration with an example switching configuration. The point of intersection of the characteristic path with the boundary of the domain is denoted by $(0, z_r(0;t^*,s^*))$ for case 1 and $(t_r(t^*,s^*), b)$ for case 2 with $z_r(t_r(t^*,s^*);t^*,s^*) = b$.
Let $z_l(t;t_r(t^*,s^*),b)$ denote the l-th characteristic path passing through $(t_r(t^*,s^*),b)$ $(l = m+1,\dots,n)$. Then, either $z_l(t;t_r(t^*,s^*),b)$ intersects the line $t = 0$ before hitting the line $s = a$ (case 2(i)) or it hits $s = a$ (case 2(ii)). The point of intersection is denoted by $(0, z_l(0;t_r(t^*,s^*),b))$ for case 2(i) and $(t_{rl}(t^*,s^*),a)$ for case 2(ii). For ease of notation, we use the notation $\tilde{s}_1 = z_r(0;t^*,s^*)$ and $\tilde{s}_2 = z_l(0;t_r(t^*,s^*),b)$ and simply t_r for $t_r(t^*,s^*)$ and t_{rl} for $t_{rl}(t^*,s^*)$.

For case 1: Integrating the $r-$th equation for any $r = 1,\dots,m$ and using $j = \sigma(t)$ in (7) we get

$$\xi_r(t^*,s^*) = \xi_r(0,\tilde{s}_1) - \int_0^{t^*}\sum_{k=1}^{n} b_{rk}^{\sigma(t)}(z_r(t))\xi_k(t,z_r(t))dt$$

---

[1] Where the dependence of $z_i(t;t^*,s^*)$ on $\sigma(\cdot)$ is omitted for notational convenience.

where we use the notation $z_r(t)$ for $z_r(t;t^*,s^*)$. Using the bound $\sum_{k=1}^{n}|b_{rk}^{j}(s)|<\epsilon$ for $r=1,\dots,m$, $s\in[a,b]$, and $j\in\mathcal{Q}$,

$$\begin{aligned}|\xi_r(t^*,s^*)| &\le \|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)}\\ &+\epsilon\int_0^{t^*}\|\boldsymbol{\xi}(t)\|_{L^\infty((a,b),\mathbb{R}^n)}dt\\ &\le\left(1+\frac{\epsilon c}{\beta}\right)\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)}\\ &+\epsilon\int_T^{t^*}\|\boldsymbol{\xi}(t)\|_{L^\infty((a,b),\mathbb{R}^n)}dt\end{aligned}$$

where the second inequality is obtained using the assumption that (25) holds on $\Omega([0,T])$. Multiplying both sides by $e^{(\beta t^*)}$, noting definition (26), and using $T\le T_{\max}$, $(t^*-t)<T_{\max}$ for all $t\in(T,t^*)$, we have

$$\begin{aligned}e^{\beta t^*}|\xi_r(t^*,s^*)| &\le C_1\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)}\\ &+C_2\int_T^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_{L^\infty((a,b),\mathbb{R}^n)}dt\end{aligned}\tag{27}$$

with $C_1=(1+\frac{\epsilon c}{\beta})e^{\beta T_{\max}}$ and $C_2=\epsilon e^{\beta T_{\max}}$.

For case 2:

$$\xi_r(t^*,s^*)=\xi_r(t_r,b)-\int_{t_r}^{t^*}\sum_{k=1}^{n}b_{rk}^{\sigma(t)}(z_r(t))\xi_k(t,z_t(t))dt$$

Using $\xi_r(t_r,b)=\sum_{l=m+1}^{n}g_{rl}^{R,j}\xi_l(t_r,b)$ with $j=\sigma(t_r)$, we have

$$\begin{aligned}|\xi_r(t^*,s^*)| &\le\sum_{l=m+1}^{n}|g_{rl}^{R,j}||\xi_l(t_r,b)|\\ &+\epsilon\int_{t_r}^{t^*}\|\boldsymbol{\xi}(t)\|_{L^\infty((a,b),\mathbb{R}^n)}dt\end{aligned}\tag{28}$$

For case $2(i)$ we have

$$\xi_l(t_r,b)=\xi_l(0,\tilde{s}_2)-\int_0^{t_r}\sum_{k=1}^{n}b_{lk}^{\sigma(t)}(z_l(t))\xi_k(t,z_l(t))dt$$

where we use the notation $z_l(t)$ for $z_l(t;t_r,b)$. Using the bound $\sum_{k=1}^{n}|b_{lk}^{j}(s)|<\epsilon$ for $l=m+1,\dots,n$, $s\in[a,b]$, and $j\in\mathcal{Q}$,

$$|\xi_l(t_r,b)|\le\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)}+\epsilon\int_0^{t_r}\|\boldsymbol{\xi}(t)\|_{L^\infty((a,b),\mathbb{R}^n)}dt$$

Substituting this bound in equation (28), and using a constant $\tilde{K}_1=\max_{j\in\mathcal{Q}}\sum_{l=m+1}^{n}|g_{rl}^{R,j}|$,

$$\begin{aligned}&|\xi_r(t^*,s^*)|\le\tilde{K}_1\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)}\\ &+\epsilon\tilde{K}_1\int_0^{t_r}\|\boldsymbol{\xi}(t)\|_{L^\infty((a,b),\mathbb{R}^n)}dt+\epsilon\int_{t_r}^{t^*}\|\boldsymbol{\xi}(t)\|_{L^\infty((a,b),\mathbb{R}^n)}dt\\ &\le K_1\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)}\\ &+K_1\epsilon\int_0^{t^*}\|\boldsymbol{\xi}(t)\|_{L^\infty((a,b),\mathbb{R}^n)}dt\\ &\le K_1\left(1+\frac{\epsilon c}{\beta}\right)\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)}\\ &+K_1\epsilon\int_T^{t^*}\|\boldsymbol{\xi}(t)\|_{L^\infty((a,b),\mathbb{R}^n)}dt\end{aligned}$$

where $K_1:=\max\{1,\tilde{K}_1\}$. Now similarly to (27), multiplying both sides by $e^{\beta t^*}$ and using $T\le 2T_{\max}$, $(t^*-t)<2T_{\max}$ for all $t\in(T,t^*)$ we have

$$\begin{aligned}e^{\beta t^*}|\xi_r(t^*,s^*)| &\le C_3\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)}\\ &+C_4\int_T^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_{L^\infty((a,b),\mathbb{R}^n)}dt\end{aligned}\tag{29}$$

with $C_3=K_1\left(1+\frac{\epsilon c}{\beta}\right)e^{2\beta T_{\max}}$ and $C_4=K_1\epsilon e^{2\beta T_{\max}}$.

For case $2(ii)$, we have

$$\xi_l(t_r,b)=\xi_l(t_{rl},a)-\int_{t_{rl}}^{t_r}\sum_{k=1}^{n}b_{lk}^{\sigma(t)}(z_l(t))\xi_k(t,z_l(t))dt$$

Using $\xi_l(t_{rl},a)=\sum_{p=1}^{m}g_{lp}^{L,j'}\xi_p(t_{rl},a)$ with $j'=\sigma(t_{rl})$, we have

$$\begin{aligned}|\xi_l(t_r,b)| &\le\sum_{p=1}^{m}|g_{lp}^{L,j'}||\xi_p(t_{rl},a)|\\ &+\epsilon\int_{t_{rl}}^{t_r}\|\boldsymbol{\xi}(t)\|_{L^\infty((a,b),\mathbb{R}^n)}dt\end{aligned}$$

Substituting this bound in equation (28), noting (24), (25), we obtain

$$\begin{aligned}|\xi_r(t^*,s^*)| &\le\theta ce^{-\beta t_{rl}}\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)}\\ &+K_1\epsilon\int_{t_{rl}}^{t^*}\|\boldsymbol{\xi}(t)\|_{L^\infty((a,b),\mathbb{R}^n)}dt\\ &\le\left(\theta+\frac{K_1\epsilon}{\beta}\right)ce^{-\beta t_{rl}}\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)}\\ &+K_1\epsilon\int_T^{t^*}\|\boldsymbol{\xi}(t)\|_{L^\infty((a,b),\mathbb{R}^n)}dt\end{aligned}$$

Multiplying both sides by $e^{(\beta t^*)}$ and noting that $T - t_{rl} \le 2T_{\max}$, we obtain

$$\begin{aligned} e^{\beta t^*}|\xi_r(t^*,s^*)| \le{}& C_5\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)} \\ &+ C_6\int_T^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_{L^\infty((a,b),\mathbb{R}^n)}dt \end{aligned} \tag{30}$$

with $C_5 = c\left(\theta+\frac{K_1\epsilon}{\beta}\right)e^{2\beta T_{\max}}$ and $C_6 = K_1\epsilon e^{2\beta T_{\max}}$. Combining (27),(29), and (30), for any $(t^*,s^*)\in\Omega([T+T_{\min}])$

$$\begin{aligned} e^{\beta t^*}|\xi_r(t^*,s^*)| \le{}& C_7\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)} \\ &+ C_6\int_T^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_{L^\infty((a,b),\mathbb{R}^n)}dt \end{aligned}$$

with $C_7 = c\left(\theta+\frac{K_1\epsilon}{\beta}\right)e^{2\beta T_{\max}}$, where we have chosen a c such that $c \ge \frac{K_1}{\theta}$.

Similar estimates can be obtained for $\xi_l(t^*,s^*)$ $(l = m+1,\dots,n)$ with $K_2 = \max_{j\in\mathcal{Q}}\{1,\sum_{p=1}^{m}|g_{lp}^{L,j'}|\}$:

$$\begin{aligned} e^{\beta t^*}|\xi_l(t^*,s^*)| \le{}& C_8\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)} \\ &+ C_9\int_T^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_{L^\infty((a,b),\mathbb{R}^n)}dt \end{aligned}$$

with $C_8 = c\left(\theta+\frac{K_2\epsilon}{\beta}\right)e^{2\beta T_{\max}}$, with $c \ge \frac{K_2}{\theta}$ and $C_9 = K_2\epsilon e^{2\beta T_{\max}}$. Thus, we get

$$\begin{aligned} \widehat{\|\boldsymbol{\xi}(t^*)\|}_{L^\infty((a,b),\mathbb{R}^n)} \le{}& C_{10}\|\bar{\xi}(t)\|_{L^\infty((a,b),\mathbb{R}^n)} \\ &+ C_{11}\int_T^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_{L^\infty((a,b),\mathbb{R}^n)}dt \end{aligned} \tag{31}$$

with $K_3 = \max\{K_1,K_2\}$, $C_{10} = c\left(\theta+\frac{K_3\epsilon}{\beta}\right)e^{2\beta T_{\max}}$, $C_{11} = K_3\epsilon e^{2\beta T_{\max}}$; finally we choose $c \ge \frac{K_3}{\theta}$. Now we define

$$\eta(t^*) := C_{10}\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)} + C_{11}\int_T^{t^*}\eta(t)dt,$$

which is equivalent to $\frac{d\eta(t)}{dt} = C_{11}\eta(t)$ with $\eta(T) = C_{10}\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)}$. Thus,

$$\begin{aligned} \eta(t^*) &= C_{10}e^{C_{11}(t^*-T)}\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)} \\ &\le C_{10}e^{2C_{11}T_{\max}}\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)} \\ &= c\left(\theta+\frac{\epsilon K_3}{\beta}\right)e^{2T_{\max}(\beta+C_{11})}\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)}. \end{aligned}$$

Note that $\widehat{\|\boldsymbol{\xi}(t^*)\|}_{L^\infty((a,b),\mathbb{R}^n)} \le \eta(t^*)$. By selecting β and ϵ such that

$$\left(\theta+\frac{K_3\epsilon}{\beta}\right)e^{2T_{\max}(\beta+K_3\epsilon e^{2\beta T_{\max}})} < 1,$$

we obtain

$$e^{\beta t^*}\|\boldsymbol{\xi}(t^*)\|_{L^\infty(a,b),\mathbb{R}^n} \le c\|\bar{\xi}\|_{L^\infty((a,b),\mathbb{R}^n)},$$

which completes the induction step. The induction basis follows directly from the combination of case 1 and case $2(i)$. □

Remark 6 *We note that the condition* (23) *in Theorem 5 is a joint condition on left and right boundary data for all pairs of modes $j, j' \in \mathcal{Q}$ whereas the conditions* (19),(20), *and* (21) *in Theorem 3 require dissipativeness of the operator $\frac{\partial A^j(s)}{\partial s} - B^j(s) - B^j(s)^\top$, the left boundary space $u_{II}(t,a) = G_L^j u_I(t,a)$ and the right boundary space $u_I(t,b) = G_R^j u_{II}(t,b)$ for all modes $j \in \mathcal{Q}$.*

Remark 7 *Note that condition* (23) *implies the following spectral radius condition to hold for the individual subsystems with $j \in \mathcal{Q}$ fixed:*

$$\rho(|G^{jj}|) < 1. \tag{32}$$

Under this assumption, classical solutions of the subsystems are known to be exponentially stable [25]. However, assumption (32) *for all $j \in \mathcal{Q}$ is not sufficient for the switched system to be exponentially stable. Note that G_L^j, G_R^j in Example 2 satisfy* (32) *for $j = 1,2$, i.e., the spectral radii of the matrices*

$G^{11} = \begin{pmatrix} 0 & 1.5 \\ 0 & 0 \end{pmatrix}$, $G^{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1.5 \end{pmatrix}$, and $G^{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ are zero but that of the matrix $G^{21} = \begin{pmatrix} 0 & 1.5 \\ 1.5 & 0 \end{pmatrix}$ is 1.5.

Nevertheless, as common for switched ODE systems, the switched system satisfying (32) in every mode j can be stabilized by switching slow enough, i. e.

Corollary 8 *(Dwell-Time) For any initial condition $\bar{\xi}$, consider switching among the systems* (5) *under the hypotheses (H_1)-(H_2) and assumption* (32) *for all $j \in \mathcal{Q}$. Then, for any $\tau \ge \bar{\tau}$ given by* (16) *assumed as dwell-time for the switching signal $\sigma(\cdot)$, the switched system is exponentially stable with respect to the L^∞-norm.*

Proof From the definition of $\bar{\tau}$ in (16) it is easy to see that if $\tau > \bar{\tau}$, then in case $2(ii)$, t_r and t_{rl} lie in the same interswitching interval and all the required estimates can be made using a $\tilde{\theta}$ defined similar to (24) but where the maximum is only taken over $j \in \mathcal{Q}$. $\square$

3.2 Stability of switched hyperbolic system in non-diagonal form

We now draw attention to not necessarily diagonal systems, i. e. switching among the systems (1) or equivalently (12). We begin with an example showing that the condition (23) for the boundaries is no longer sufficient for the switched system to be absolutely exponentially stable.

Example 9 *Consider the following 2-mode switched system ($\mathcal{Q} = \{1, 2\}$).*

$$\begin{aligned} &\frac{\partial u}{\partial t} + A^j \frac{\partial u}{\partial s} = 0, \\ A^1 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}&, \quad A^2 = \begin{pmatrix} -1 & -4 \\ 0 & 1 \end{pmatrix}, \\ u^2(t,0) &= \begin{cases} \frac{3}{2} u^1(t,0), & j = 1 \\ \frac{-3}{4} u^1(t,0), & j = 2 \end{cases} \\ u^1(t,1) &= \begin{cases} \frac{1}{4} u^2(t,1), & j = 1 \\ 0, & j = 2, \end{cases} \end{aligned} \tag{33}$$

for an alternating switching signal $\sigma(\cdot)$ with switching times $\tau_k = 0.5k$ where $k = 0, 1, 2, 3, \dots$ and $\sigma(\tau_0) = 1$, $\sigma(\tau_1) = 2$, $\sigma(\tau_2) = 1$ and so on. Let the initial condition be $\bar{u}(s) = 1, \quad s \in (0, 1)$. It can be easily seen that this example satisfies H_1^*, H_2, and H_3 and that

$$S_1 A^1 S_1^{-1} = S_2 A^2 S_2^{-1} = \operatorname{diag}(-1, 1) := \Lambda, \tag{34}$$

where $S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $S_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Defining the characteristic variables $\xi(t,s) = (\xi^1(t,s), \xi^2(t,s))^\top$ as $\xi = S_j u$ for $j \in \{1, 2\}$, the switched system (33) in characteristic variables becomes:

$$\begin{aligned} \frac{\partial \xi}{\partial t} + \Lambda \frac{\partial \xi}{\partial s} = 0, \\ \xi^2(t,0) = \frac{3}{2} \xi^1(t,0), \quad \xi^1(t,1) = \frac{1}{4} \xi^2(t,1), \end{aligned} \tag{35}$$

and the initial condition in characteristic variables become $\bar{\xi}(s) = 1, \quad s \in (0, 1)$. The characterizing matrix for both $j \in \{1, 2\}$ has spectral radius $0.6124 < 1$. It is easy to observe that for the system (33), the solution at all switching times τ_k is constant in $s \in (0, 0.5]$ (resp. in $s \in (0.5, 1)$) denoted by $(v_I^1(\tau_k), v_I^2(\tau_k))^\top$ (resp. by $(v_{II}^1(\tau_k), v_{II}^2(\tau_k))^\top$). Thus, the value of the solution at times τ_k is

$$u(\tau_k, s) = \xi(\tau_k, s) = \begin{cases} (v_I^1(\tau_k), v_I^2(\tau_k))^\top & s \in [0, 0.5] \\ (v_{II}^1(\tau_k), v_{II}^2(\tau_k))^\top & s \in (0.5, 1). \end{cases} \tag{36}$$

where $v_I^1(\tau_0) = v_I^2(\tau_0) = v_{II}^1(\tau_0) = v_{II}^2(\tau_0) = 1$. The solution at τ_{k+1} is

$$\xi(\tau_{k+1}, s) = \begin{cases} \begin{pmatrix} v_{II}^1(\tau_k) + 3 v_I^1(\tau_k) \\ \frac{3}{2} v_I^1(\tau_k) \end{pmatrix} & s \in (0, 0.5] \\ \begin{pmatrix} \frac{1}{4} v_{II}^2(\tau_k) + 2 v_I^2(\tau_k) \\ v_I^2(\tau_k) \end{pmatrix} & s \in (0.5, 1). \end{cases} \tag{37}$$

and the solution at τ_{k+2} is

$$\xi(\tau_{k+2}, s) = \begin{cases} \begin{pmatrix} v_{II}^1(\tau_{k+1}) - 3 v_I^1(\tau_{k+1}) \\ \frac{3}{2} v_I^1(\tau_{k+1}) \end{pmatrix} & s \in (0, 0.5] \\ \begin{pmatrix} \frac{1}{4} v_{II}^2(\tau_{k+1}) - 2 v_I^2(\tau_{k+1}) \\ v_I^2(\tau_{k+1}) \end{pmatrix} & s \in (0.5, 1). \end{cases} \tag{38}$$

The quantities in equations (36) and (38) evolve as

$$\begin{pmatrix} v_I^1(\tau_{k+2}) \\ v_I^2(\tau_{k+2}) \\ v_{II}^1(\tau_{k+2}) \\ v_{II}^2(\tau_{k+2}) \end{pmatrix} = \begin{pmatrix} -9 & 2 & -3 & 0.25 \\ 4.5 & 0 & 1.5 & 0 \\ -3 & 0.25 & 0 & 0 \\ 1.5 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_I^1(\tau_k) \\ v_I^2(\tau_k) \\ v_{II}^1(\tau_k) \\ v_{II}^2(\tau_k) \end{pmatrix}$$

which is an unstable system. Thus, $\|\xi(\tau_k, \cdot)\|_{L^\infty((0,1),\mathbb{R}^2)}$ and $\|u(\tau_k, \cdot)\|_{L^\infty((0,1),\mathbb{R}^2)}$ is not bounded as $k \to \infty$.

We recall the fact that a set of diagonizable matrices are simultaneously diagonizable if (and only if) they commute. As an easy consequence of simultaneous diagonalizability of all subsystems in (1), we have the following.

Corollary 10 *The statements of Theorem 3, Theorem 5 and Corollary 8 in Section 3.1 also hold for switching among the non-diagonal systems (1) if the advective velocity matrix functions commute pairwise, i. e. $A^j(s) A^{j'}(s) = A^{j'}(s) A^j(s)$ for all $j, j' \in \mathcal{Q}$ and for all $s \in [a, b]$.*

Proof Under the simultaneous diagonalizability, we have $S_j(\cdot) \equiv S(\cdot)$. Now the equations (13) and (12) respectively give $\boldsymbol{\xi}(t) = T^{j_k}(t - \tau_k) \dots T^{j_0}(\tau_1 - \tau_0) \bar{\xi}(\cdot)$ and $\mathbf{u}(t) = S^{-1}(\cdot) \boldsymbol{\xi}(t)$ for $t \in (\tau_k, \tau_{k+1})$. Results from Section 3.1 can now be extended to this setting. $\square$

For general, possibly not commuting matrices A^j, the difficulty comes with a (sufficiently tight) growth bound for the semigroups in (12) in terms of the boundary data in order to obtain absolutely exponential stability as in [21].

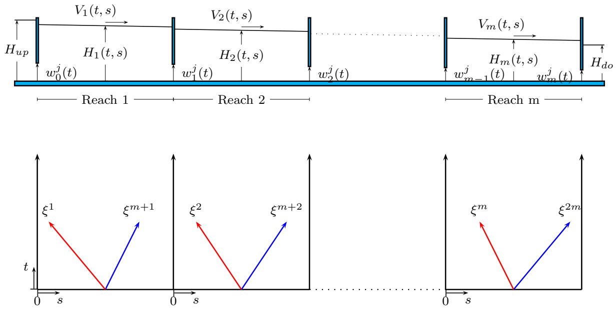

Fig. 4. (a) Cascade of canals operated by multi-mode underflow sluice gates. (b) Characteristic variables characterizing each reach.

4 Application for linearized Saint-Venant equations

We apply the stability results to water flow in a cascade of m canal reaches as depicted in Figure 4 (a). Consider a setting in which a supervisory controller selects between a set of boundary control actions applied at the underflow sluice gates with corresponding gate openings w_i^j for reach i in mode j. Theorem 5 can be applied to investigate the stability of linearized dynamics to a steady-state flow in such a $m-$reach cascade of open channels for a situation in which boundary control actions switch between a number of modes.

The flow of water in reach i is characterized by velocity $V_i(t,s)$ and elevation $H_i(t,s)$. For horizontal, prismatic canals with rectangular cross-section, frictionless walls and normalized length, the flow, under gravity g, satisfies the Saint-Venant equations [17]

$$\frac{\partial}{\partial t}\begin{pmatrix} H_i \\ V_i \end{pmatrix} + \begin{pmatrix} V_i & H_i \\ g & V_i \end{pmatrix} \frac{\partial}{\partial s}\begin{pmatrix} H_i \\ V_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{39}$$

for $i = 1, \ldots, m$, each defined on the domain $\{(t,s) : 0 \le t < \infty, 0 \le s \le 1\}$. Following [4], let the initial data be given by $H_i(0,s)$, $V_i(0,s)$ and the boundary conditions modeling decentralized feedback control actions in mode j together with flow conservation for each reach i be given by

$$\begin{aligned} &f_1^j(w_0^j(t), H_{\mathrm{up}}, H_1(t,0), V_1(t,0)) = 0 \\ &f_i^j(w_i^j(t), H_i(t,1), H_{i+1}(t,0), V_i(t,1)) = 0 \\ &f_m^j(w_m^j(t), H_m(t,1), H_{\mathrm{do}}, V_m(t,1)) = 0 \\ &H_i(t,1)V_i(t,1) - H_{i+1}(t,0)V_{i+1}(t,0) = 0 \end{aligned}$$

where H_{up}, H_{do} are the (known) up and down stream water levels.

Assume that under constant gate openings $\bar{w}_i$ and constant $H_{\mathrm{up}}, H_{\mathrm{do}}$, each reach attains a uniform steady state $(\bar{H}_i, \bar{V}_i)$ such that $H_{\mathrm{do}} < \bar{H}_m < \ldots < \bar{H}_1 < H_{\mathrm{up}}$ and $\bar{H}_1\bar{V}_1 > 0$. Using $v_i(x,t) = V_i(x,t) - \bar{V}_i$ and $h_i(x,t) = H_i(x,t) - \bar{H}_i$, the linearized model can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} h_i \\ v_i \end{pmatrix} + \begin{pmatrix} \bar{V}_i & \bar{H}_i \\ g & \bar{V}_i \end{pmatrix} \frac{\partial}{\partial s}\begin{pmatrix} h_i \\ v_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{40}$$

with initial conditions $h_i(0,\cdot)$, $v_i(0,\cdot)$ for $i = 1, \ldots, m$. The traditional Riemann coordinate change [17] $\xi_i(t,s) = h_i(t,s) + v_i\sqrt{\bar{H}_i/g}$, $\xi_{m+i}(t,s) = h_i(t,s) - v_i\sqrt{\bar{H}_i/g}$ leads to a diagonal system:

$$\frac{\partial}{\partial t}\begin{pmatrix} \xi_i \\ \xi_{m+i} \end{pmatrix} + \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_{m+i} \end{pmatrix} \frac{\partial}{\partial s}\begin{pmatrix} \xi_i \\ \xi_{m+i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{41}$$

with $\lambda_i = (\sqrt{g\bar{H}_i} - \bar{V}_i)$ and $\lambda_{m+i} = (\sqrt{g\bar{H}_i} + \bar{V}_i)$.

Under sub-critical flow, the eigenvalues satisfy $\lambda_i < 0 < \lambda_{m+i}$. For the system of $m-$canal reaches, equation (41) can be written in the form

$$\partial_t \xi + \Lambda \partial_s \xi = 0, \tag{42}$$

where $\xi = (\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_{2m})^\top$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{2m})$ (see Figure 4 (b)). Moreover, setting $\xi_I = (\xi_1, \ldots, \xi_m)$, $\xi_{II} = (\xi_{m+1}, \ldots, \xi_{2m})$ and taking into account the coordinate transformation while assuming sufficient regularity of f_i^j, the boundary conditions in linearized form for each j can be rewritten as

$$\xi_{II}(t,0) = G_L^j \xi_I(t,0) \quad \xi_I(t,1) = G_R^j \xi_{II}(t,1) \tag{43}$$

with appropriately defined jacobians G_L^j, G_R^j (for details on the derivation for an explicit control law f_i^j see [4]).

Our results from Section 3.1 provide a set of sufficient conditions for solutions of (42)-(43) to decay for any admissible supervisory control action, e. g. as to pursue superior objectives. In this context, the dwell-time results appear to be conventional, taking into account the multiscale peculiarity of the modeling.

5 Final remarks

We present first results on stability of infinite dimensional systems that undergo switching among a set of hyperbolic PDEs that may differ in the system matrix function and/or boundary conditions. For the case when constituent PDEs are in diagonal form, we derive sufficient conditions for asymptotic stability under arbitrary switching signals. These results extend the well-known sufficient conditions of [8] and [25]. For the case when the system matrix functions are not diagonal, these results hold when they are jointly diagonalizable. This results in a commutativity condition that has an analogue in the switched ODE literature [18].

It should have become clear that, although the switching signal was taken as global, the results apply for switching the boundary conditions or system matrices individually by introducing appropriate auxiliary modes, this is just a matter of notational convenience. So our results can be applied to study the stability of linearized Saint-Venant equations that model water flow in open-channels under abruptly changing boundary conditions and operating regimes. In this context, it should also be noted that Theorem 3 allows for an (abrupt) change of sign for the eigenvalues in the system matrices. Thus we provide a comparatively easy but mathematically sophisticated model to treat stability in the context of transcritical flow regimes, e. g. in combination with sequential linearization.

Lastly, our results motivate further study of stability of PDE system that undergo switching in time, in particular, future direction of work should include extension of a Lyapunov theory for switched PDE systems.

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