Chapter 5

Problems in Plastic Flow and Collapse I
Theories and “Exact” Solutions

Introduction

In Chapter 4 the concepts of plastic flow and plastic collapse were regarded as essentially equivalent, representing a state in which a body continues to deform under constant applied forces. In practical applications, however, the two concepts have quite different meanings. Plastic collapse describes undesirably large deformations of an already formed body (a structure) that result from excessive forces; the calculation of collapse loads of simple structures was studied in Section 4.1. The concept of plastic flow, on the other hand, is usually applied to the deliberate forming of a mass of solid (such as metal or clay) into a desired shape through the application of appropriate forces.\footnote{The plastic flow of soil constitutes an exception, since substantial movement of a soil mass supporting a building or forming an earth dam is generally regarded as failure.} It is remarkable that these two large classes of problems, of fundamental importance in mechanical and civil engineering, can be attacked by the same methodology — the theory of rigid–perfectly plastic bodies, with the help of the theorems of limit analysis. A particularly extensive body of theory, filling entire books, exists for problems of plane strain; a summary of the theory, with some applications, is presented in Section 5.1. In Section 5.2 we deal with the plastic collapse of circular plates.

Apart from plastic collapse, collapse of an elastic–plastic body may also be due to structural instability. Such collapse (e.g., the buckling of a column) may begin when the body is still fully elastic, and plastic deformation occurs as a part of post-buckling behavior. Buckling that follows yield is covered by the theory of plastic instability, which is treated in Section 5.3.
Section 5.1 Plane Problems

In Section 4.3 we found the stress field in a pressurized elastic–perfectly plastic hollow cylinder in a state of plane strain by solving the equilibrium equations together with, on the one hand, the compatibility condition in the elastic region and, on the other hand, the yield criterion in the plastic region, and by satisfying the boundary conditions on the outer and inner surfaces and continuity conditions at the elastic-plastic boundary. The solution is valid for all pressures up to the ultimate pressure $p_U$.

When the ultimate pressure is reached, the tube becomes fully plastic. The compatibility and continuity conditions then become irrelevant. The equilibrium equation (4.3.18) and the yield criterion now constitute equations for the two unknown stress components $\sigma_r$ and $\sigma_\theta$, which may be solved so that the traction boundary conditions are satisfied. For both the Mises and Tresca yield criteria (which are equivalent in plane plastic flow), the solution produces the stresses given by Equations (4.3.30) with $c = b$.

Since the fully plastic tube problem is formulated entirely in terms of the stresses, it is often said to be *statically determinate*, though in a looser sense than that of Section 4.1, since it is not equilibrium ("statics") alone that determines the stress field, but the yield criterion as well. The fully plastic torsion problem of 4.2.3 can similarly be characterized as statically determinate, since once again there is one nontrivial equilibrium equation which, together with the yield criterion, can be used to determine the two unknown stress components subject to the traction boundary conditions.

The same notion of static determinacy can be applied, in principle, to a body of arbitrary shape that is assumed to be undergoing plane plastic deformation, or to be in a state of plane stress: there are, in general, three unknown stress components ($\sigma_{11}$, $\sigma_{22}$, $\sigma_{12}$ in Cartesian coordinates) and two plane equilibrium equations, plus the yield criterion. However, static determinacy in this sense is effective in producing a unique stress field only in special cases, namely those in which (a) only traction boundary conditions are relevant, and (b) the entire body must become plastic for unrestricted plastic flow to occur. As we saw in the beam problems studied in Section 4.5, it is in general possible to have unrestricted flow in a plastic region that occupies only a part of the body, the rest of the body remaining in the elastic range and hence behaving as though it were rigid (recall, from 3.5.1, the vanishing of the elastic strain rates at incipient plastic flow).

Even when the aforementioned conditions (a) and (b) are satisfied, there may occur situations in which the traction boundary are not by themselves sufficient to choose between two possible stress tensors at a point (see Figure 5.1.4, which is discussed later).

Since, in the general case, some of the boundary conditions may be kinematic (velocity boundary conditions), it becomes necessary to find a kine-
matically admissible velocity field such that the strain rates derived from it obey the flow rule. Consequently, the stress and velocity problems are coupled. In the special cases where a unique stress field can be found directly, then the velocity field can be found afterwards, as was done with the warping of a fully plastic rectangular shaft (see 4.2.3). In the axisymmetric plane-strain problem, the only velocity component is the radial velocity $v$, and the flow rule is equivalent to the incompressibility constraint

$$\frac{dv}{dr} + \frac{v}{r} = 0,$$

which may be solved to give $v(r) = v(a)(a/r)$.

If, in addition, elastic regions remain in the course of plastic flow, then a solution of the problem requires the determination of the elastic-plastic boundary and of a stress field in the elastic regions which is continuous with the plastic stress field at the boundary. While this was accomplished for the beam problems of Section 4.5, it is in general an exceedingly difficult task. In many problems, the main objective is to find the load that produces plastic flow or collapse, and a complete elastic–plastic solution is not necessary to achieve this objective: we know from the theorems of limit analysis (Section 3.5) that the correct critical load is obtained from a plastically and statically admissible stress field and a kinematically admissible velocity field that is associated with it in the plastic region. Thus a rigid-plastic boundary may be established on a purely kinematic basis, and the plastic stress field needs to be extended into the rigid region only so that it is statically and plastically admissible. With this extension, the solution becomes a complete rigid–plastic solution, usually known simply as a complete solution (Bishop [1953]).

A systematic method of determining stress fields and associated velocity fields in perfectly plastic bodies obeying the Mises (or Tresca) yield criterion in plane strain was developed in the 1920s by Prandtl, Hencky, Mises and others, and generalized by Mandel [1962] to include other yield criteria and plane stress. This method, generally known as slip-line theory, is discussed in the next subsection. Some applications are presented in succeeding subsections.

### 5.1.1. Slip-Line Theory

#### Shear Directions

A convenient way to establish the necessary relations for the stress field in a plastic region is with the help of the definitions $n = \frac{1}{2}(\sigma_1 + \sigma_2)$ and $r = \frac{1}{2}|\sigma_1 - \sigma_2|$, as given in 3.3.4, in conjunction with the yield condition (3.3.6). We introduce the Mohr’s circle relations

$$\sigma_{11} = n + r \sin 2\theta, \quad \sigma_{22} = n - r \sin 2\theta, \quad \sigma_{12} = -r \cos 2\theta,$$  \hspace{1cm} (5.1.1)
where \( \theta \) is the angle from the \( x_1 \) axis to one of the principal shear directions, namely the one along which the maximum shear stress \( r \) is directed to the left when one is facing the outer normal (see Figure 5.1.1). This direction will be called the second shear direction, and a line having this direction locally everywhere will be called a second shear line. The other shear direction (shear line) will be called the first.

When equations (5.1.1) are substituted into the equilibrium equations

\[
\sigma_{11,1} + \sigma_{12,2} = 0, \quad \sigma_{12,1} + \sigma_{22,2} = 0,
\]

the resulting equations are, upon substitution of \( r = h(n) \),

\[
[1 + h'(n) \sin 2\theta]n_{,1} - h'(n) \cos 2\theta n_{,2} + 2r(\theta_1 \cos 2\theta + \theta_2 \sin 2\theta) = 0,
\]

\[
-h'(n) \cos 2\theta n_{,1} + [1 - h'(n) \sin 2\theta]n_{,2} + 2r(\theta_1 \sin 2\theta - \theta_2 \cos 2\theta) = 0.
\]

Equations (5.1.2) constitute a pair of nonlinear partial differential equations for \( n, \theta \). A useful method of numerical solution for these equations is the method of characteristics.

**Method of Characteristics**

In order to understand how this method can be used to solve the system, it is simpler to consider first a single first-order partial differential equation of the form

\[
Av_{,1} + Bv_{,2} = C,
\]

where \( A, B \) and \( C \) are functions of \( x_1, x_2 \) and \( v \). The equation can be
multiplied by an infinitesimal increment $dx_1$ and rewritten as

$$A(v_{,1} dx_1 + \frac{B}{A} v_{,2} dx_1) = C dx_1.$$  

The quantity in parentheses becomes a perfect differential $dv = (v_{,1}) dx_1 + (v_{,2}) dx_2$ if $dx_2 = m dx_1$, where $m = B/A$. The direction defined by $dx_2/dx_1 = m$ is called a characteristic direction, and a curve that is everywhere tangent to a characteristic direction is known as a characteristic curve or simply a characteristic. Along such a curve, then,

$$\frac{dx_1}{A} = \frac{dx_2}{B} = \frac{dv}{C}.$$  

If $v$ is known at one point of a characteristic curve, then $dv$ can be calculated for a neighboring point on this curve, and continuing this operation allows $v$ to be determined at all points on the curve. If $v$ is known at all points of a curve $\Sigma$ that is nowhere tangent to a characteristic, then its values may be calculated along all the characteristics that intersect $\Sigma$.

If two characteristics emanating from different points of $\Sigma$ should intersect at some point $Q$ of the $x_1x_2$-plane, then in general two different values of $v$ will be obtained there. The point $Q$ is therefore the locus of a discontinuity in $v$.

Consider, now, a curve $\Sigma$ that is nowhere tangent to a characteristic, and suppose that a discontinuity (jump) in $v_{,1}$ or $v_{,2}$ occurs at a point $P$ of $\Sigma$. Since the only information necessary to determine the characteristic curve through $P$ and the values of $v$ along this curve is the value of $v$ at $P$, the directional derivative of $v$ along the characteristic will have the same value on either side of it. Consequently, if any jump in $v_{,2}$ or $v_{,1}$ is propagated through the $x_1x_2$-plane, it must occur across the characteristic through $P$.

In order to apply the method of characteristics to a system of several first-order partial differential equations, the number of real characteristic directions at each point must equal the number of unknown variables. The system is then called hyperbolic. The system (5.1.2) is consequently hyperbolic if it has two real characteristic directions. It is called parabolic if there is one such direction, and elliptic if there are none.

**Characteristics of Equations (5.1.2)**

In order to determine whether Equations (5.1.2) constitute a hyperbolic system, we begin by solving them for $n_{,1}, n_{,2}$ in terms of $\theta_{,1}, \theta_{,2}$:

$$n_{,1} = \frac{-2r}{1 - h'^2} (\theta_{,1} \cos 2\theta + \theta_{,2} \sin 2\theta - h' \theta_{,2}),$$

$$n_{,2} = \frac{-2r}{1 - h'^2} (\theta_{,1} \sin 2\theta - \theta_{,2} \cos 2\theta + h' \theta_{,1}).$$
Note that the solutions break down when $|h'(n)| = 1$. Assuming that this breakdown does not occur, we attempt to determine the characteristics by assuming that, along a characteristic, $dn = \lambda d\theta$. Now

\[
\begin{align*}
  dn &= n_1 \, dx_1 + n_2 \, dx_2 \\
  &= \frac{-2r}{1 - h'^2} \left\{ \theta_1 \left[ \cos 2\theta \, dx_1 + (\sin 2\theta + h') \, dx_2 \right] + \theta_2 \left[ (\sin 2\theta - h') \, dx_1 - \cos 2\theta \, dx_2 \right] \right\} \\
  &= \lambda \, d\theta \\
  &= \theta_1 (\lambda \, dx_1) + \theta_2 (\lambda \, dx_2).
\end{align*}
\]

Equating the coefficients of $\theta_1$ and $\theta_2$, respectively, in the second and fourth lines leads to the system of equations

\[
\begin{align*}
  (1 - h'^2) \lambda + 2r \cos 2\theta \, dx_1 + 2r (\sin 2\theta + h') \, dx_2 &= 0, \\
  2r (\sin 2\theta - h') \, dx_1 + [(1 - h'^2) \lambda - 2r \cos 2\theta] \, dx_2 &= 0,
\end{align*}
\]

which constitute a second-order eigenvalue problem. The characteristic equation is

\[
(1 - h'^2)^2 \lambda^2 - 4r^2 \cos^2 2\theta - 4r^2 \sin^2 2\theta + 4r^2 h'^2,
\]

yielding the eigenvalues

\[
\lambda = \pm \frac{2r}{\sqrt{1 - h'^2}}.
\]

The roots are real if and only if $|h'| \leq 1$; the problem is hyperbolic if $|h'| < 1$, parabolic when $|h'| = 1$, and elliptic when $|h'| > 1$.

Referring to the examples of the yield condition (3.3.6) discussed following its formulation, we note that the plane-stress problem for the Mises criterion is hyperbolic only when $|n| < 3k/2$, parabolic when $|n| = 3k/2$, and elliptic when $|n| > 3k/2$; this last condition occurs when

\[
\frac{1}{2} < \frac{\sigma_1}{\sigma_2} < 2.
\]

For the Tresca criterion, the problem is hyperbolic when $|n| < k$ and parabolic when $|n| \geq k$; the elliptic case does not arise.

In plane strain, on the other hand, $h' \equiv 0$ for both criteria with the associated flow rule when the elastic strains can be neglected, and therefore the problem is hyperbolic throughout the plastic domain. The same is true for the Mohr–Coulomb criterion with the appropriate (not necessarily associated) flow rule, since $h' = -\sin \phi$, where $\phi$ is the angle of internal friction. More generally, for any yield criterion given by Equation (3.3.6) with $|h'| < 1$, it is convenient to define a variable angle of internal friction, $\phi(n)$, by $\sin \phi(n) = -h'(n)$; this is just the inclination of the Mohr envelope with respect to the $\sigma$-axis at the point of tangency. The eigenvalues $\lambda$ are correspondingly given by $\pm 2r \sec \phi$. 
From the trigonometric identity
\[ \tan(a + b) = \frac{\sin 2a + \sin 2b}{\cos 2a + \cos 2b} \]
we may derive the directions of the eigenvectors. The eigenvector corresponding to \( \lambda = 2r \sec \phi \) is given by \( dx_2/dx_1 = -\cot(\theta - \frac{1}{2} \phi) \), and will be called the first or \( \alpha \) characteristic, while the eigenvector corresponding to \( \lambda = -2r \sec \phi \) is given by \( dx_2/dx_1 = \tan(\theta + \frac{1}{2} \phi) \), and will be called the second or \( \beta \) characteristic. In the case of the Mises and Tresca criteria in plane strain (the classical case), the characteristic directions coincide with the shear directions and are orthogonal. In general the characteristics of the two families intersect at an angle \( \frac{1}{2} \pi \pm \phi \); they coalesce into a single family when \( \phi = \frac{1}{2} \pi \), that is, in the parabolic case.

Defining the dimensionless variable \( \omega \) by
\[ \omega = \int \frac{\cos \phi(n)}{2h(n)} dn, \]
we may write the characteristic relations, following Mandel [1962], as
\[
\begin{align*}
d\omega &= d\theta \quad \text{along a first characteristic}, \\
d\omega &= -d\theta \quad \text{along a second characteristic}.
\end{align*}
\]
(5.1.3)

Note that \( \omega = n/2k \) in the classical case, and
\[ \omega = -\frac{\cot \phi}{2} \ln \left( 1 - \frac{n}{c} \tan \phi \right) \]
for the Mohr–Coulomb material, yielding the preceding value in the limit as \( \phi \to 0 \), with \( c = k \).

If we introduce a curvilinear coordinate system \( \alpha, \beta \) such that first and second characteristics are given respectively by \( \beta = \text{constant} \) and \( \alpha = \text{constant} \), then we can write the canonical equations
\[
\frac{\partial}{\partial \alpha} (\omega - \theta) = 0, \quad \frac{\partial}{\partial \beta} (\omega + \theta) = 0,
\]
which have the general solution
\[ \omega = \xi(\alpha) + \eta(\beta), \quad \theta = \xi(\alpha) - \eta(\beta), \]
\( \xi \) and \( \eta \) being arbitrary functions.

In the following discussion of the geometric properties of the characteristic network we shall limit ourselves to the case with \( \phi \) constant — that is, the Mohr–Coulomb criterion — so that a change in \( \theta \) is also a change in the direction of the characteristic. This case includes the classical case, \( \phi = 0 \), and
the characteristic network is then called a Hencky–Prandtl network. Among its properties are the following:

1. Suppose that one \( \alpha \) characteristic is straight, that is, for a given \( \beta \), \( \partial \theta / \partial \alpha = 0 \). It follows that \( \xi'(\alpha) = 0 \), and consequently all the \( \alpha \) characteristics are straight. Obviously, the same result holds for \( \beta \) characteristics.

2. Consider a pair of \( \alpha \) characteristics, defined by \( \beta \) and \( \beta' \), respectively, and a pair of \( \beta \) characteristics defined by \( \alpha \) and \( \alpha' \), the points of intersection being labeled \( A = (\alpha, \beta) \), \( B = (\alpha', \beta) \), \( C = (\alpha, \beta') \), and \( D = (\alpha', \beta') \), as in Figure 5.1.2. It follows from the solution above that

\[
\chi_{AC} = \theta(\alpha, \beta) - \theta(\alpha, \beta') = \eta(\beta') - \eta(\beta),
\]

\[
\chi_{BD} = \theta(\alpha', \beta) - \theta(\alpha', \beta') = \eta(\beta') - \eta(\beta),
\]

and consequently the two angles \( \chi_{AC} \) and \( \chi_{BD} \) are equal. It can similarly be shown that \( \chi_{AB} \) equals \( \chi_{CD} \). This result is due to Hencky [1923] and is known as Hencky’s theorem. In words: the angle formed by the tangents of two given characteristics of one family at their points of intersection with a characteristic of the other family does not depend on the choice of the intersecting characteristic of the other family.

3. Now take \( (\alpha', \beta') \) in Figure 5.1.2 infinitesimally close to \( (\alpha, \beta) \). If \( R_{\beta}(\alpha, \beta) \) denotes the radius of curvature of the \( \beta \) characteristics at \( (\alpha, \beta) \) and if \( ds_\alpha \) and \( ds_\beta \) denote infinitesimal arc lengths along the \( \alpha \) and \( \beta \) characteristics, respectively, then

\[
ds_{\beta(AC)} = R_{\beta}(\alpha, \beta)\chi_{AC} = [R_{\beta}(\alpha', \beta) + ds_\alpha]\chi_{BD}.
\]

Since \( R_{\beta}(\alpha', \beta) = R_{\beta}(\alpha, \beta) + dR_{\beta} \), it follows that \( dR_{\beta} = -ds_\alpha \) along an \( \alpha \) characteristic. Similarly, \( dR_{\alpha} = ds_\beta \) along a \( \beta \) characteristic. This result is due to Prandtl [1923].

**Traction Boundary Conditions**

Traction boundary-value problems may be of three types, with the construction of characteristics corresponding to each type shown in Figure 5.1.3.
Problem 1. The boundary is nowhere tangent to a characteristic.

Problem 2. The boundary is composed of characteristics of both families.

Problem 3. The boundary is of mixed type.

![Diagram](a)
![Diagram](b)
![Diagram](c)

Figure 5.1.3. Traction boundary-value problems: (a) Problem 1; (b) Problem 2; (c) Problem 3.

If the relation between $\omega$ and $n$ is invertible, then the state of stress at a point is determined by $(\omega, \theta)$. Referring to Figure 5.1.4, we note that, along an arc whose normal forms an angle $\chi$ with the $x_1$-axis, the normal stress, shear stress and transverse (interior) normal stress are respectively given by

$$
\sigma = n + r \sin 2(\theta - \chi), \quad \tau = r \cos 2(\theta - \chi), \quad \sigma' = n - r \sin 2(\theta - \chi),
$$

(5.1.4)

where $n$ and $r$ are determined by $\omega$. If the arc forms a part of the boundary, however, then at most $\sigma$ and $\tau$ will be given there (traction boundary conditions); $\sigma'$ can then have either of the values $\sigma \pm 2\sqrt{r^2 - \tau^2}$. Usually the right value of $\sigma'$ can be chosen by physical intuition.

![Diagram](sigma_tau)

Figure 5.1.4. Stresses at a boundary.

In the classical case, the two choices for $\sigma'$ give the respective explicit expressions for $\theta$ and $\omega$

$$
\theta = \chi \pm \frac{1}{2} \cos^{-1} \frac{\tau}{k}, \quad \omega = \frac{\sigma}{2k} \pm \frac{1}{2} \sqrt{1 - \frac{\tau^2}{k^2}}.
$$
Stress Discontinuities

An arc such as that of Figure 5.1.4 may also be located in the interior of the plastic domain and be part of a line of stress discontinuity. To satisfy equilibrium, \( \sigma \) and \( \tau \) must be continuous across such a line, but \( \omega \) (and therefore \( n \)) and \( \theta \) are discontinuous, so that the directions of the characteristics of each family change abruptly. In the general case \( r \) is also discontinuous, but in the classical case \( r \) equals \( k \) and is therefore continuous; \( \sigma' \) then takes each of the two possible values \( \pm 2\sqrt{k^2 - \tau^2} \) on either side of the discontinuity line. Note that there is no discontinuity if and only if \( |\tau| = k \), that is, if the arc is along a characteristic (which in the classical case is a shear line). When \( |\tau| < k \), \( \theta \) changes by \( \cos^{-1}(\tau/k) \). It can be seen from a Mohr's-circle construction that the line of stress discontinuity must bisect the angles formed by the characteristics of each family on either side of it. It can also be seen that if the discontinuity line is thought of as the limit of a narrow zone of continuous but rapid change in \( \sigma' \) while \( \sigma \) and \( \tau \) remain constant, all the intermediate Mohr's circles must be of radius less than \( k \), showing that this zone is elastic and that the discontinuity line is therefore the remnant of an elastic zone (just like the ridge lines in the torsion problem). As a result of this property, it was shown by Lee [1950] that a line of stress discontinuity acts like an inextensible but perfectly flexible filament.

If the two regions separated by a line of stress discontinuity are denoted 1 and 2, and if the inclination of the line is \( \chi \), then \( \sigma \) and \( \tau \) as given by the first two Equations (5.1.4) are continuous across this line. Specializing to the classical case, with \( n = 2\omega k \) and \( r = k \), leads to

\[
2\omega_1 + \sin 2(\theta_1 - \chi) = 2\omega_2 + \sin 2(\theta_2 - \chi),
\]

\[
\cos 2(\theta_1 - \chi) = \cos 2(\theta_2 - \chi).
\]

These equations may be solved for \( \theta_2 \) and \( \omega_2 \) in terms of \( \theta_1 \) and \( \omega_1 \), yielding the jump conditions due to Prager [1948]:

\[
\theta_2 = 2\chi - \theta_1 \pm n\pi, \quad \omega_2 = \omega_1 \pm \sin 2(\theta_1 - \chi), \quad (5.1.5)
\]

\( n \) being an integer, and the appropriate sign being taken as indicated by the problem.

It was shown by Winzer and Carrier [1948] that if several straight lines of stress discontinuity separating domains of constant stress meet at a point, then these lines must number at least four. Winzer and Carrier [1949] also discussed stress discontinuities between fields of variable stress.

The preceding arguments can be carried over from the classical to the general case (see Salençon [1977]). In particular, the jump conditions may be written directly in terms of the variables in Equations (5.1.4),

\[
n_1 + r_1 \sin 2(\theta_1 - \chi) = n_2 + r_2 \sin 2(\theta_2 - \chi),
\]

\[
r_1 \cos 2(\theta_1 - \chi) = r_2 \cos 2(\theta_2 - \chi). \quad (5.1.6)
\]
**Velocity Fields**

If a traction boundary-value problem is solved by constructing a characteristic network, as described above, over a part of the region representing the body, then the loading forms an upper bound to that under which plastic flow becomes possible, since, as will be shown below, a kinematically admissible velocity field can then be found. As mentioned before, the exact flow load is found when the stress field can be extended in a statically and plastically admissible manner into the rigid region. The following discussion will be limited to the classical case; for a more general discussion, see, for example, Salençon [1977].

The equations governing the velocity components \( v_1, v_2 \) are found by combining the associated flow rule,

\[
\frac{\dot{\varepsilon}_{11}}{\sigma_{11} - \sigma_{22}} = \frac{\dot{\varepsilon}_{22}}{\sigma_{22} - \sigma_{11}} = \frac{\dot{\varepsilon}_{12}}{2\sigma_{12}},
\]

with the strain-rate–velocity relations

\[
\dot{\varepsilon}_{11} = v_{1,1}, \quad \dot{\varepsilon}_{22} = v_{2,2}, \quad \dot{\varepsilon}_{12} = \frac{1}{2}(v_{1,2} + v_{2,1})
\]

and with Equations (5.1.1) to obtain

\[
v_{1,1} + v_{2,2} = 0, \quad v_{1,2} + v_{2,1} - 2\cot 2\theta v_{2,2} = 0. \tag{5.1.7}
\]

The characteristics of Equations (5.1.7) are found by assuming \( dv_1 = \lambda dv_2 \).

Thus

\[
dv_1 = v_{1,1} \, dx_1 + v_{1,2} \, dx_2 = v_{2,1} (-dx_2) + v_{2,2} (2\cot 2\theta \, dx_2 - dx_1) \\
= \lambda \, dv_2 = v_{2,1} (\lambda \, dx_1) + v_{2,2} (\lambda \, dx_2),
\]

so that the characteristic directions are the eigenvectors of the system

\[
\lambda \, dx_1 + dx_2 = 0,
\]

\[
dx_1 + (\lambda - 2\cot 2\theta) \, dx_2 = 0.
\]

The eigenvalues \( \lambda \) are the roots of

\[
\lambda^2 - 2\lambda \cot 2\theta - 1 = 0,
\]

namely,

\[
\lambda = \cot 2\theta \pm \csc 2\theta = \left\{ \begin{array}{c}
\cot \theta \\
-\tan \theta.
\end{array} \right.
\]

For \( \lambda = \cot \theta \) we have \( dx_2/dx_1 = -\cot \theta \) (i.e., the first shear direction) while for \( \lambda = -\tan \theta \) we have \( dx_2/dx_1 = \tan \theta \) (i.e., the second shear direction).
We see therefore that the characteristics of the velocity equations are the same as those of the stress equations. The characteristic relations are thus

\[
\begin{align*}
\frac{dv_1}{dv_2} &= \cot \theta & \text{along an } \alpha \text{ characteristic,} \\
\frac{dv_1}{dv_2} &= -\tan \theta & \text{along a } \beta \text{ characteristic.}
\end{align*}
\]

A plane whose coordinates are \(v_1\) and \(v_2\) is known as the hodograph plane, and a diagram in this plane showing the velocity distribution is called a hodograph. If \(P\) and \(Q\) are two neighboring points in the \(x_1x_2\)-plane (the physical plane) lying on the same shear line, and if \(P'\) and \(Q'\) are the points in the hodograph plane representing the respective velocities, then as shown by Geiringer [1951] and Green [1951], it follows from the characteristic relations that the line element \(P'Q'\) is perpendicular to \(PQ\). Consequently the Hencky–Prandtl properties apply to the hodograph as well. A rigid region, if it does not rotate, is represented by a single point in the hodograph plane.

It is also instructive to express the characteristic relations in terms of the velocity components along the characteristic directions, given respectively by

\[
\begin{align*}
v_\alpha &= v_1 \sin \theta - v_2 \cos \theta \\
v_\beta &= v_1 \cos \theta + v_2 \sin \theta.
\end{align*}
\]

The relations were derived by Geiringer [1931] and are known as the Geiringer equations:

\[
\begin{align*}
\frac{dv_\alpha}{dv_\beta} &= v_\beta \, d\theta & \text{along an } \alpha \text{ characteristic,} \\
\frac{dv_\beta}{dv_\alpha} &= -v_\alpha \, d\theta & \text{along a } \beta \text{ characteristic.}
\end{align*}
\]

It can be seen that these relations express the condition that the longitudinal strain rates \(\dot{\varepsilon}_{11}\) and \(\dot{\varepsilon}_{22}\) vanish when the \(x_1\) and \(x_2\) axes coincide locally with the characteristic directions, in other words, that the shear lines are inextensible. This result also follows directly from the flow rule, since with respect to such axes we have \(\sigma_{11} - \sigma_{22} = 0\).

It must be remembered, however, that the flow rule actually gives only the ratios among the strain rates, and the preceding result must strictly be written as \(\dot{\varepsilon}_{11}/\dot{\varepsilon}_{12} = \dot{\varepsilon}_{22}/\dot{\varepsilon}_{12} = 0\). Another interpretation of this result is that \(\dot{\varepsilon}_{12}\) is infinite, meaning that either \(v_{1,2}\) or \(v_{2,1}\) is infinite. This happens if the tangential velocity component is discontinuous across a characteristic (the normal component must be continuous for material continuity), that is, if slip occurs. The characteristics are thus the potential loci of slip and are therefore also called slip lines. Kinematically admissible velocity fields with discontinuities across slip lines are often used in the construction of solutions. In particular, slip may occur along a characteristic forming the boundary between the plastic and rigid regions.

If slip occurs along an \(\alpha\) characteristic, with the tangential velocity having the values \(v_\alpha\) and \(v^*\alpha\) on either side of it, then, since \(v_\beta\) has the same value on both sides, an application of Equation (5.1.8)\(_1\) along the two sides of the slip line gives \(dv_\alpha = dv^*\alpha\), or \(d(v^*\alpha - v_\alpha) = 0\). An analogous result applies to a \(\beta\) characteristic. Thus the discontinuity in the tangential velocity
remains constant along a slip line. It follows further that the curves in the hodograph plane that form the images of the two sides of a line of velocity discontinuity are parallel.

5.1.2. Simple Slip-Line Fields

A great many practical problems can be solved by means of slip-line fields containing straight slip lines. As we have seen, if one slip line (characteristic) of a given family is straight, then all the slip lines of that family must be straight. Families of straight slip lines, as can be seen in Figure 5.1.5, may be of three types: (a) parallel, (b) meeting at a point, and (c) forming an envelope.

Figure 5.1.5. Families of straight slip lines: (a) constant-state field; (b) centered fan; (c) noncentered fan with envelope.

(a) If all the slip lines of one family are straight and parallel, then those of the other family must be likewise. Since \( \theta \) is constant, it follows from Equations (5.1.3) that \( \omega \) is constant as well, and therefore that the state of stress is uniform. A region in which the slip lines are of this type is called a region of constant state; this term is taken from wave-propagation theory and is not strictly applicable here, because, while the stress components are constant, the velocity components are not necessarily so; Equations (5.1.8), with \( d\theta = 0 \), yield the solutions \( v_\alpha = f(\beta) \), \( v_\beta = g(\alpha) \). The functions \( f \) and \( g \) are arbitrary except as constrained by boundary conditions.

(b) If the slip lines of one family are straight and meet at a point, then those of the other family must be concentric circular arcs. Such a system of slip lines is called a centered fan. A number of problems may be solved by inserting a centered fan between two regions of constant state, in such a way that the bounding radial lines of the fan are also the bounding parallel lines of the constant-state regions. \( \omega \) is constant along all the straight lines, while along the circular arcs of the fan, \( d\omega = \pm d\theta \), depending on whether the arcs are \( \alpha \) or \( \beta \) characteristics. If the \( \theta \) difference between the two constant-state regions is \( \Delta \theta \), then this is just the angle subtended by the bounding lines of the fan, and \( \Delta \omega = \pm \Delta \theta \).

(c) An envelope of slip lines is also called a limiting line, and a family
Chapter 5 / Problems in Plastic Flow and Collapse

Chapter 5 / Problems in Plastic Flow and Collapse

of straight slip lines forming an envelope is called a noncentered fan; the
envelope is called the base curve of the fan. A limiting line cannot be in the
interior of the plastic region, and therefore must form a part of either the
boundary of the body or of the rigid-plastic boundary. Other properties of
limiting lines are discussed by Prager and Hodge [1951], Section 25.

Some Applications

We now consider some simple applications of slip-line fields consisting of
constant-state regions and centered fans, illustrated in Figure 5.1.6. These
results are of considerable importance in soil mechanics, where they are used
to study the stability of slopes and the carrying capacity of foundations made
of clays for which the hypothesis of constant shear strength (the undrained
strength discussed in 2.3.1) can be justified.

(a) Consider, first, a wedge of angle $2\gamma$ with a uniform pressure on one
side and no traction on the other. A possible shear-line net consists of two
regions of constant state, separated by a line of stress discontinuity bisecting
the wedge. With the principal stresses $2k - p, -p$ on one side and $0, -2k$
on the other, continuity of the normal stress across the discontinuity line is
possible only if $p = 4k \sin^2 \gamma$. This value is therefore a lower bound to the
pressure causing incipient plastic flow. As can be seen in Figure 5.1.6(a),
when the wedge is acute ($\gamma < \frac{1}{4} \pi$) a velocity field may be constructed such
that regions $ABC$ and $AEF$ slip along the slip lines $AB$ and $AF$, respec-
tively, while $ACDE$ flows perpendicular to the stress-discontinuity line $AD$,
so that slip also occurs along $AC$ and $AE$. The material below $BAF$ may
be rigid.

(b) When the wedge is obtuse ($\gamma > \frac{1}{4} \pi$) no such velocity field is possible.
On the other hand, it is now possible to insert a centered fan of angle $2\gamma - \frac{1}{2} \pi$
between two constant-state regions, producing the pressure $p = 2k(1 + 2\gamma -
\frac{1}{2} \pi)$, which exceeds the previously obtained lower bound for all $\gamma > \frac{1}{4} \pi$.

(c) Now consider a truncated wedge with pressure on the top face. At
each corner we can construct a plastic zone consisting of a centered fan
between two triangular regions of constant state, and plastic flow can occur
when the two plastic regions meet, so that the top face can have a downward
velocity. The angle subtended by each fan is just $\gamma$, so that $p = 2k(1 + \gamma)$.

(d) The limit of the preceding case as $\gamma \to \frac{1}{2} \pi$ represents a half-plane
carrying a rigid block, and therefore the limiting pressure on the interface
between the half-plane and the block is $k(2 + \pi)$. This result was obtained
by Prandtl [1920] by assuming a single triangular region of constant state
under the block, with a centered fan on either side and another constant-
state region outside each fan [see Figure 5.1.6(f)]. Prandtl’s solution was
criticized by Hill, who pointed out that, since the elastic solution of the
problem leads to infinite stresses at the corners of the block, plastic zones
must be there from the outset. As the pressure is increased, these zones will
Figure 5.1.6. Simple slip-line fields: (a) acute wedge; (b) obtuse wedge; (c) truncated wedge loaded on the top edge; (d) rigid block on a half-plane, Hill solution; (e) finite truncated wedge, discontinuous stress field; (f) rigid block on a half-plane, Prandtl solution.
grow until they meet, as in case (c).

In the last two examples, no stress field outside the plastic regions has been presented, so that the resulting pressures must be regarded as upper bounds only. For example (c), Drucker and Chen [1968] have shown how to construct a statically admissible stress field, leading to a lower bound equal to the upper bound on the pressure. The following example shows that for a finite truncated wedge, a lower pressure than that obtained in (c) above can be found.

(e) A solution for a finite truncated wedge, uniformly loaded on its top and bottom faces, is based on a fully plastic stress distribution, with the four triangular constant-state regions separated by the stress-discontinuity lines $OA, \ldots, OD$, which bisect the angles at the corners. In order for these bisectors to meet at one point, the ratio of the bottom to the top face of the trapezoid must be $(1 + \sin \gamma)/(1 - \sin \gamma)$. Because of symmetry, only the right half of the wedge need be considered. Since the flank $BC$ is traction-free, region 2 is in a state of simple compression parallel to $BC$, and the values of $\theta$ and $\omega$ there are $\theta_2 = \gamma + \pi/4$ and $\omega_2 = -\frac{1}{2}$, respectively. In region 1, by symmetry, $\theta = \theta_1 = \pi/4$. By Equation (5.1.5)$_2$, $\omega_1 = -\frac{1}{2} - \sin \gamma$. Equation (5.1.1)$_2$ then gives, in region 1,

$$\sigma_y = -2k(1 + \sin \gamma) = -p,$$

where $p$ is the pressure on the top face; note that this value is less than that obtained in (c) above. The pressure on the bottom face can similarly be found as $q = 2k(1 - \sin \gamma)$. It can be seen that the aforementioned geometric restriction is necessary for equilibrium.

Problems with Circular Symmetry

In classical problems with axial symmetry, the slip lines are along the shear directions and therefore at 45° to the radial and tangential directions. They are therefore given by logarithmic spirals, $r \propto e^{\pm \theta}$. These slip lines can be used to construct velocity fields in hollow prisms with a circular bore under internal pressure. In Figure 5.1.7, for example, an axisymmetric stress field — the same as in the hollow cylinder under internal pressure — is assumed in the region inside the largest circle, with a vanishing stress field outside this circle. Plastic flow is assumed to occur only in the curved triangular regions bounded by the slip lines that meet the sides of the square at their midpoints. The remaining regions move diagonally outward as rigid bodies. A statically admissible stress field and an associated kinematically admissible velocity field are thus found, and the pressure must be that for the cylinder, namely, $p = 2k \ln b/a$. 
5.1.3. Metal-Forming Problems

A number of problems representing metal-forming processes can be solved approximately by means of the theory of plane plastic strain, if the metal is idealized as a material that is rate-independent and rigid–perfectly plastic, and if the thermal stresses that result from the temperature gradients induced by the forming process can be neglected. In a metal-forming problem — unlike a structural problem — unrestricted plastic flow is the desired condition. The solution is intended to furnish the smallest applied force under which the metal will flow, rather than the largest load under which the structure will not collapse. For this reason an upper bound to the force is a "safe" answer.

Metal-forming processes that closely approximate plane-strain conditions (and for which slip-line theory can be used to generate upper bounds on the forming forces) include forging, indentation, and cutting of wide strips, as well as continuous processes such as extrusion, drawing and rolling. In problems representing the latter category of processes, it is not the initiation of the process that is studied, but a state of steady plastic flow in which a large amount of plastic deformation has already taken place, and the stress and velocity fields are taken as constant in time in an Eulerian sense, much as in steady-flow problems of fluid mechanics.

An extensive bibliography of slip-line fields for metal-forming processes can be found in Johnson, Sowerby and Venter [1982]. Many examples are also to be found in Johnson and Mellor [1973], Chapters 11 and 14, and Chakrabarty [1987], Chapters 7 and 8. Only a few selected problems will be
treated here.

**Indentation**

The problem of the half-plane carrying a rigid block, illustrated in Figure 5.1.6(d), can also be interpreted as describing the beginning of indentation of a half-plane by a flat punch. The Hill solution, which requires slip between the plastic zones and the punch, implies smooth contact. In the Prandtl solution, on the other hand, the triangular region directly under the punch moves rigidly downward with it, corresponding to rough contact. This material in this region is often called *dead metal*.

The solution shown in Figure 5.1.6(c) for the truncated wedge of half-angle $\gamma$ with uniform pressure on its top face can be adapted to the problem of indentation by a flat indenter at the bottom of a flat trench (Figure 5.1.8) if $\gamma$ is replaced by $\pi - \gamma$. The indentation pressure is thus $p = 2k(1 + \pi - \gamma)$.

![Figure 5.1.8. Flat indenter at the bottom of a flat trench.](image)

A solution for the frictionless indentation of a half-plane by an acute wedge-shaped indenter of half-angle $\alpha$ ($\alpha < \pi/4$) was proposed by Hill, Lee and Tupper [1947]. This problem is one of *pseudo-steady plastic flow*, in which the geometry of the slip-line field (and therefore the stress and velocity fields) changes as penetration proceeds, but in a geometrically similar manner — that is, the angles remain the same, and only the scale changes. As shown in Figure 5.1.9, the slip-line field covers Zone 1. Zone 3 is that in which the material is elastic and therefore treated as rigid, while the intermediate Zone 2 contains material that has yielded but is restrained from moving. As can be seen, the solution allows for the piling up of material (the formation of a “coronet”) about the indenter, although in practice such piling up is observed only in work-hardening materials. The angle $\theta$ subtended by the centered fan between the two constant-state triangles can be determined from the wedge half-angle $\alpha$ by means of the condition that the volume of piled-up material (shown as shaded in the figure) equals the volume of that portion of the indenter that has penetrated the work (shown crosshatched). The slip-line field is equivalent to that in Figure 5.1.6(b), with $\theta = 2\gamma - \pi/2$, and therefore the contact pressure $q$ is

$$q = 2k(1 + \theta).$$
The indenter force, at a given state of indentation, is thus

\[ F = 4k(1 + \theta)AB. \]

Because of geometric similarity, the indenter force is consequently proportional to indentation depth.

**Forging and Cutting**

The simultaneous application of identical flat punches to a strip of finite thickness, Figure 5.1.10(a), may be used to model forging, with the bottom punch representing the anvil and the top punch the forging tool. Similarly, the cutting of a strip of metal with a wirecutter-like tool can be described as the simultaneous indentation by a pair of identical wedge-shaped indenters located opposite each other [Figure 5.1.10(b)]. Because of symmetry, in each case only the top half of the strip needs to be considered, and the middle plane may be regarded as a frictionless foundation. The solution of both problems was studied by Hill [1953].

**Figure 5.1.10.** Forging and cutting: (a) forging; (b) cutting.

For the cutting problem Hill showed that when the plastic region has not yet reached the foundation, the slip-line field is the same as for the
semi-infinite domain. When, however, the plastic region extends through the thickness of the strip, a different mode of deformation takes over: piling up ceases, and the material on either side of the plastic region moves rigidly outward; the slip-line field is shown in Figure 5.1.11(a).

![Slip-line fields: (a) forging; (b) cutting.](Figure 5.1.11)

For the forging problem, the slip-line field is shown in Figure 5.1.11(b). A triangular dead-metal region attaches itself to the punch, and indentation proceeds as in the cutting problem, provided that \( h < 8.74a \), where \( h \) is the half-thickness of the strip and \( a \) the half-width of the punch. When \( h = 8.74a \), it can be shown that the punch pressure is \( p = 2k(1 + \pi/2) \), as for the semi-infinite domain. It follows that for \( h > 8.74a \) the zone of plastic deformation does not go through the strip and the pressure remains at this value.

**Drawing and Extrusion**

Drawing and extrusion are processes in which a billet of material is forced to flow through a die shaped to produce the required cross-section. In drawing, as the name suggests, the material is pulled. Extrusion involves pushing. In *direct* extrusion the die is stationary with respect to the container holding the billet, and a ram moves in the container, pushing the billet outward with the help of a pressure pad. In *reverse* extrusion the container is closed at one end, and the die is pushed inside the container. The three processes are shown in Figure 5.1.12.

The technologically important applications of these processes are predominantly three-dimensional — drawing produces wire, and extrusion is used to make lightweight structural shapes, trim and the like. In the absence of three-dimensional solutions, however, the solutions of the corresponding plane problems provide qualitative information on the nature of the plastic regions and hence allow estimates for the required forces.

Figure 5.1.13(a) illustrates a solution due to Hill [1948c] describing frictionless extrusion through a square die with 50% reduction. The slip-line field consists the two centered fans \( OAB \) and \( O'A'B' \); because of symmetry,
only $OAB$ need be considered. Since the exit slip line $OA$ is a line of constant stress, $\sigma_{11}$ must vanish identically on it in order for the extruded metal to its left to be in equilibrium, and $\sigma_{12} = 0$ because the line forms an angle of $45^\circ$ with the $x$-axis. The yield criterion requires $|\sigma_{22}| = 2k$, and, since the sheet is being compressed, it follows that $\sigma_{22} = -2k$. Equations (5.1.1), with $n = 2k\omega$ and $r = k$, accordingly require that $\omega = -\frac{\pi}{2}$ and $\theta = \pi/4$ on $OA$.

The characteristic relations can now be used to determine the state along $AB$. Since the fan subtends $90^\circ$, $\theta = 3\pi/4$, and Equation (5.1.3)$_2$ shows that $\omega = \frac{1}{2}(1 + \pi)$ there. By Equations (5.1.1), then, we have $\sigma_{11} = -(2 + \pi)k$, $\sigma_{22} = -\pi k$, and $\sigma_{12} = 0$ on $AB$. The average value of $-\sigma_{11}$ along $OAB$ is thus equal to the extrusion pressure,

$$p = \left(1 + \frac{\pi}{2}\right)k.$$

A statically admissible extension of the stress field into the rigid region due to Alexander [1961] shows this to be the exact pressure, not merely an upper bound. The stress field on $OB$ is extended into the dead-metal region $ABC$, while that on $OA$ is extended into $OAP$; the extruded metal to the left of $AP$ is stress-free, so that $AP$ is a line of stress discontinuity. The extension to the right of the arc $OB$ is achieved analytically.

A simple slip-line field for a reduction of $\frac{2}{3}$ is shown in Figure 5.1.13(b),

Figure 5.1.12. Drawing and extrusion: (a) drawing; (b) direct extrusion; (c) reverse extrusion.
and leads to an extrusion pressure of

\[ p = \frac{4}{3} \left( 1 + \frac{\pi}{2} \right) k. \]

Figures 5.1.14(a) and (b) illustrate both drawing and extrusion through a tapered die. If the container walls in the extrusion problem are smooth, then the slip-line fields are identical if the die angle \( \alpha \) and the reduction ratio \( r \) are the same; the stress fields in the regions covered by the slip-line field differ only by a hydrostatic stress.

A particularly simple slip-line field due to Hill and Tupper [1948], valid for a smooth die when \( r = 2 \sin \alpha / (1 + 2 \sin \alpha) \), is shown in Figure 5.1.14(c). In the extrusion problem, \( \omega \) and \( \theta \) on the exit slip line OA are the same as in the preceding problem, and therefore their values in the constant-state region ABC are \( \omega = -\frac{1}{2}(1 + 2\alpha) \), \( \theta = \pi/4 + \alpha \). The normal pressure on AC is

\[ q = (1 - 2\omega)k = 2(1 + \alpha)k, \]

and the tangential stress there is zero. For equilibrium, the extrusion pressure \( p \) must be

\[ p = rq = \frac{4(1 + \alpha)\sin 2\alpha}{1 + 2\sin 2\alpha} k. \]
The corresponding drawing problem is solved by superposing a hydrostatic tension equal in magnitude to this pressure. The drawing stress therefore has the same value as the extrusion pressure.

For further solutions of metal-forming problems using slip-line theory, see the aforementioned references by Johnson and Mellor [1973], Johnson, Sowerby and Venter [1982], and Chakrabarty [1987]. Additional plane problems, in which complete solutions are not available, are discussed in Section 6.1 in the context of limit analysis.

**Exercises: Section 5.1**

1. For a Hencky–Prandtl network in which the values of $\omega - \theta$ on any two neighboring first characteristics and the values of $\omega + \theta$ on any two neighboring second characteristics differ by the same small constant, show that the diagonals of the network are lines of constant $\omega$ or of constant $\theta$ (use Figure 5.1.2).

---

1Finite-element methods for metal-forming problems are treated by Kobayashi, Oh, and Altan [1989].
2. Show that when four straight lines of stress discontinuity meet on an axis of symmetry of stress field and separate four regions of constant stress, as in Figure 5.1.6(e), the angles $\angle AOD$ and $\angle BOC$ must be right angles.

3. Show that any velocity field that is associated with the stress field of the preceding exercise represents rigid-body motion.

4. Derive the Geiringer equations (5.1.8).

5. Find the differential equations for the velocity field in plane plastic flow in a standard Mohr–Coulomb material. Determine the characteristics of the velocity field.

6. Show that in classical problems of plane plastic flow with axial symmetry, the slip lines are given in polar coordinates by $r \propto e^{\pm \theta}$.

7. Show that the relation between $\theta$ and $\alpha$ in the slip-line field of Figure 5.1.9 is

$$\alpha = \frac{1}{2} \left[ \theta + \cos^{-1} \tan \left( \frac{\pi}{4} - \frac{\theta}{2} \right) \right].$$

8. The slip-line field of Figure 5.1.6(c) may be regarded as representing a stage in the squashing of an originally pointed wedge of half-angle $\alpha$ (greater than $\gamma$) by a lubricated flat plate.

   (a) Using geometry and volume constancy, find the relation between $\alpha$ and $\gamma$.
   
   (b) Find the relation between the applied force and the distance moved by the plate.
   
   (c) Determine the smallest value of $\alpha$ for which the solution is valid.

9. Discuss how the slip-line field of Figure 5.1.6(c) can be used to study the necking of a symmetrically notched tension specimen.

10. Discuss the velocity fields associated with the slip-line fields of Figures 5.1.13(a) and (b).

**Section 5.2  Collapse of Circular Plates**

The goal of this section is derivation of collapse loads for axisymmetrically loaded circular plates made of a perfectly plastic material obeying the Tresca criterion. For such plates, complete solutions exist in closed form, and they are treated in 5.2.3. For the collapse loads of plates without circular symmetry, limit analysis must be used to obtain estimates, and this is done in
Chapter 6. The introduction to plate theory given in 5.2.1 is general, and not limited to circular plates. Similarly, the presentation of elastic relations and yield criteria at the beginning of 5.2.2 and 5.2.3, respectively, is general, but solutions will be given for axisymmetric problems only.

5.2.1. Introduction to Plate Theory

Derivation of Plate Equilibrium Equations

A plate may be defined as a solid body occupying in the undeformed configuration the region \( A \times [-h/2, h/2] \), that is, the set of points \( (x_1, x_2, x_3) \mid (x_1, x_2) \in A, -h/2 \leq x_3 \leq h/2 \), where \( A \) is a closed domain in the \( x_1x_2 \)-plane bounded by a simple closed curve \( C \) (we are assuming no holes in the plate), with \( h \) considerably smaller than the typical dimension of \( A \). The plane \( x_3 = 0 \) is called the middle plane of the plate. The outward normal unit vector to \( C \) in the \( x_1x_2 \)-plane has components \( \nu_\alpha (\alpha = 1, 2) \), and the counterclockwise tangential unit vector has components \( t_\alpha \).

We shall approach the study of the mechanics of plates by a combined use of the three-dimensional equilibrium equations and of virtual work. The plate equilibrium equations will be derived directly from the former. First, we define the stress resultants as follows:

\[
N_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} \, dx_3 \quad \text{(membrane forces)},
\]

\[
Q_\alpha = \int_{-h/2}^{h/2} \sigma_{\alpha 3} \, dx_3 \quad \text{(shear forces)},
\]

\[
M_{\alpha\beta} = -\int_{-h/2}^{h/2} x_3 \sigma_{\alpha\beta} \, dx_3 \quad \text{(moments)}.
\]

The surface loads are

\[
p_\alpha = \sigma_{\alpha 3} \big|_{-h/2}^{h/2} + \int_{-h/2}^{h/2} f_\alpha \, dx_3,
\]

\[
q = \sigma_{33} \big|_{-h/2}^{h/2} + \int_{-h/2}^{h/2} f_3 \, dx_3,
\]

\[
m_\alpha = -x_3 \sigma_{\alpha 3} \big|_{-h/2}^{h/2} - \int_{-h/2}^{h/2} x_3 f_\alpha \, dx_3,
\]

where \( f \) is the body force per unit volume.

Distinguishing the \( x_3 \)-coordinate from the \( x_\alpha (\alpha = 1, 2) \), we write the local equilibrium equations

\[
\sigma_{\alpha\beta,\beta} + \sigma_{\alpha 3,3} + f_\alpha = 0, \quad (5.2.1)
\]

\[
\sigma_{\alpha 3,\alpha} + \sigma_{33,3} + f_3 = 0. \quad (5.2.2)
\]
Integrating these equations through the thickness and performing integration by parts where necessary yields

\[ N_{\alpha\beta,\beta} + p_{\alpha} = 0 \] (5.2.3)

and

\[ Q_{\alpha,\alpha} + q = 0. \] (5.2.4)

When Equations (5.2.1) are multiplied by \( x_3 \) and then integrated through the thickness, the result is

\[ M_{\alpha\beta,\beta} + Q_{\alpha} + m_{\alpha} = 0. \] (5.2.5)

We can eliminate \( Q_{\alpha} \) between Equations (5.2.4) and (5.2.5), and obtain

\[ M_{\alpha\beta,\alpha\beta} = q - m_{\alpha,\alpha}. \] (5.2.6)

Equations (5.2.3) and (5.2.4–5) or (5.2.6) are the plate equilibrium equations, the former for in-plane or membrane forces and the other for bending forces. Note that the two modes of behavior — in-plane deformation and bending — are statically uncoupled. In the elementary theory they are also kinematically uncoupled, and therefore can be studied separately.

**Displacement Assumptions and Virtual Work**

The elementary displacement model for plate behavior is described by the following displacement field:

\[ u_{\alpha}(x_1, x_2, x_3) = \bar{u}_{\alpha}(x_1, x_2) - x_3 w_{,\alpha}(x_1, x_2), \quad u_3(x_1, x_2, x_3) = w(x_1, x_2); \]

here the \( \bar{u}_{\alpha} \) are the in-plane displacements and \( w \) is the deflection. It follows that \( \varepsilon_{i3} = 0, \ i = 1, 2, 3, \) so that \( \sigma_{ij} \delta \varepsilon_{ij} = \sigma_{\alpha\beta} \delta \varepsilon_{\alpha\beta}, \) and \( \varepsilon_{\alpha\beta} = \bar{\varepsilon}_{\alpha\beta} - x_3 w_{,\alpha\beta}, \) where \( \bar{\varepsilon}_{\alpha\beta} = \frac{1}{2}(\bar{u}_{\alpha,\gamma} + \bar{u}_{\beta,\alpha}). \)

It is important to check whether the assumption of infinitesimal strain is valid. The Green–Saint-Venant strain tensor (Section 1.2) has the in-plane components

\[ E_{\alpha\beta} = \varepsilon_{\alpha\beta} + \frac{1}{2} w_{,\gamma,\alpha} u_{,\gamma,\beta} + \frac{1}{2} w_{,\alpha} w_{,\beta}. \]

If we neglect the contributions of the in-plane displacements \( \bar{u}_{\alpha}, \) then the right-hand side reduces to

\[ -x_3 w_{,\alpha\beta} + \frac{1}{2} x_3^2 w_{,\alpha\gamma} w_{,\beta\gamma} + \frac{1}{2} w_{,\alpha} w_{,\beta}. \]

If \( \delta \) is a typical deflection and \( l \) a typical dimension of \( A, \) then the first term is of order \( h\delta/l^2, \) the second of order \( (h\delta/l^2)^2, \) and the third of order \( (\delta/l)^2. \) While the second term is negligible in comparison to the third whenever \( h/l \) is sufficiently small, as is normal in plate theory, for the third term to be
negligible in comparison to the first term it is necessary for the deflection to be small compared to the plate thickness. Otherwise, the Green–Saint-Venant strain tensor must be used, given in general by

\[ E_{\alpha\beta} = \tilde{E}_{\alpha\beta} - x_3 w_{\alpha\beta}, \]

with

\[ \tilde{E}_{\alpha\beta} = \tilde{\varepsilon}_{\alpha\beta} + \frac{1}{2} w_{\alpha} w_{\beta}. \quad (5.2.7) \]

Equation (5.2.7) will be used in the next section when the buckling of plates is studied.

Under the hypothesis of infinitesimal strain, the internal virtual work becomes

\[ \delta W_{\text{int}} = \int_A (N_{\alpha\beta} \delta \tilde{\varepsilon}_{\alpha\beta} + M_{\alpha\beta} \delta w_{\alpha\beta}) \, dA. \]

This equation may be rewritten as

\[ \delta W_{\text{int}} - \delta W^{(1)}_{\text{ext}} = \delta W^{(2)}_{\text{ext}}, \]

where \( \delta W^{(1)}_{\text{ext}} \) denotes the part of the external virtual work due to the body force and the surface tractions on the planes \( x_3 = \pm h/2 \), and \( \delta W^{(2)}_{\text{ext}} \) is that due to applied forces and moments along the edge. The first part accordingly is given by

\[ \delta W^{(1)}_{\text{ext}} = \int_A \left[ \int_{-h/2}^{h/2} f_i \delta u_i \, dx_3 + (\sigma_{33} \delta u_i) \right]_{-h/2}^{h/2} dA. \]

Now

\[ f_i \delta u_i = f_\alpha \delta \tilde{u}_\alpha - x_3 f_\alpha \delta w_\alpha + f_3 \delta w \]

and

\[ (\sigma_{33} \delta u_i) \big|_{-h/2}^{h/2} = \sigma_{33} \big|_{-h/2}^{h/2} \delta \tilde{u}_\alpha - (x_3 \sigma_{33}) \big|_{-h/2}^{h/2} \delta w_\alpha + \sigma_{33} \big|_{-h/2}^{h/2} \delta w, \]

so that

\[ \delta W^{(1)}_{\text{ext}} = \int_A (p_\alpha \delta \tilde{u}_\alpha + q \delta w + m_\alpha \delta w_\alpha) \, dA. \]

Since \( N_{\alpha\beta} \) is symmetric, it follows that \( N_{\alpha\beta} \delta \tilde{\varepsilon}_{\alpha\beta} = N_{\alpha\beta} \delta \tilde{u}_{\alpha\beta} \), and

\[ \begin{align*}
\delta W_{\text{int}} - \delta W^{(1)}_{\text{ext}} &= \int_A [(N_{\alpha\beta} \delta \tilde{u}_{\alpha\beta} - p_\alpha \delta \tilde{u}_\alpha) + (M_{\alpha\beta} \delta w_{\alpha\beta} - m_\alpha \delta w_\alpha - q \delta w)] \, dA.
\end{align*} \]

Now

\[ N_{\alpha\beta} \delta \tilde{u}_{\alpha\beta} = (N_{\alpha\beta} \delta \tilde{u}_\alpha)_{;\beta} - N_{\alpha\beta;\beta} \delta \tilde{u}_\alpha, \]

and

\[ M_{\alpha\beta} \delta w_{\alpha\beta} - m_\alpha \delta w_\alpha = (M_{\alpha\beta} \delta w_\alpha)_{,\beta} - [(M_{\alpha\beta,\beta} + m_\alpha) \delta w]_{,\alpha} + (M_{\alpha\beta;\alpha} + m_{\alpha;\alpha}) \delta w, \]
so that upon applying the two-dimensional divergence theorem we obtain

$$\delta W_{\text{int}} - \delta W_{\text{ext}}^{(1)} = \oint_C \left[ \nu_\beta N_{\alpha\beta} \delta \bar{u}_\alpha + \nu_\beta M_{\alpha\beta} \delta w_{,\alpha} + \nu_\alpha (M_{\alpha\beta,\beta} + m_\alpha) \delta w \right] ds$$

$$- \int_A \left[ (N_{\alpha\beta} + p_\alpha) \delta \bar{u}_\alpha + (M_{\alpha\beta,\alpha\beta} + m_{\alpha\alpha} - q) \delta w \right] dA.$$  

(5.2.8)

However, by the equilibrium equations (5.2.3) and (5.2.6) the area integral vanishes. In the remaining contour integral, the expression in parentheses may be replaced by $-Q_\alpha$ as a result of Equation (5.2.5). Moreover, the three functions $w, w_{,1}$ and $w_{,2}$ are not independent on $C$, because, if $w_{,\alpha}$ is decomposed as $w_{,\alpha} = \nu_\alpha \partial w / \partial n + t_\alpha \partial w / \partial s$, then the normal derivative $\partial w / \partial n$ (which represents the local rotation of the plate) can be prescribed independently of $w$, but the tangential derivative $\partial w / \partial s$ is entirely determined by $w$. Performing the aforementioned decomposition and defining $M_n = \nu_\alpha \nu_\beta M_{\alpha\beta}$ (the normal bending moment), $M_{nt} = \nu_\alpha t_\beta M_{\alpha\beta}$ (the twisting moment), and $Q_n = \nu_\alpha Q_\alpha$, we obtain

$$\delta W_{\text{int}} - \delta W_{\text{ext}}^{(1)} = \oint_C \left( \nu_\beta N_{\alpha\beta} \delta \bar{u}_\alpha + M_n \frac{\partial \delta w}{\partial n} + M_{nt} \frac{\partial \delta w}{\partial s} + Q_n \delta w \right) ds.$$

By integration by parts,

$$\oint_C M_{nl} \frac{\partial \delta w}{\partial s} ds = - \oint_C \frac{\partial M_{nt}}{\partial s} \delta w ds,$$

and therefore

$$\delta W_{\text{int}} - \delta W_{\text{ext}}^{(1)} = \oint_C \left( \nu_\beta N_{\alpha\beta} \delta \bar{u}_\alpha + M_n \frac{\partial \delta w}{\partial n} + M_{nt} \frac{\partial \delta w}{\partial s} + Q_n \delta w \right) ds,$$

where $V_n = Q_n - \partial M_{nt} / \partial s$ is the effective shear force along the edge. A graphic illustration of the equivalence between a varying twisting moment and a distributed transverse force may be seen in Figure 5.2.1.

![Figure 5.2.1. Effective shear force along a plate edge.](image)

Finally, let the applied in-plane forces along the edge be $F_{\alpha}^{a}$, the applied bending moment (acting about the tangent to $C$) $M_{n}^{a}$, and the applied transverse force $V_{n}^{a}$, all per unit length. Then

$$\delta W_{\text{ext}}^{(2)} = \oint_C \left( F_{\alpha}^{a} \delta \bar{u}_\alpha + M_n \frac{\partial \delta w}{\partial n} + V_n \delta w \right) ds.$$
so that the boundary conditions are

either \( \nu_\beta N_\alpha = F^a_\alpha \) or \( \bar{u}_\alpha \) prescribed,

either \( M_n = M^a_n \) or \( \frac{\partial w}{\partial n} \) prescribed,

either \( V_n = V^a_n \) or \( w \) prescribed.

Leaving aside the in-plane forces and displacements, we see that at every point of the edge two conditions must be specified. For example, along a clamped edge the conditions are \( w = 0 \) and \( \partial w / \partial n = 0 \); along a simply supported edge, \( w = 0 \) and \( M_n = 0 \); and along a free edge, \( V_n = 0 \) and \( M_n = 0 \). The condition \( V_n = 0 \) was first derived by Kirchhoff, and consequently the theory of plates that has thus far been outlined is known as **Kirchhoff plate theory**. In the original plate theory formulated by Sophie Germain, a free edge was assumed to be subject to the three boundary conditions \( Q_n = 0, M_{nt} = 0, \) and \( M_n = 0 \), resulting in an improperly posed boundary-value problem for elastic plates.

In the present treatment the principle of virtual work was used to derive the boundary conditions that are consistent with the displacement model adopted, while the equilibrium equations (5.2.3)-(5.2.6) were derived from the three-dimensional ones — that is, they were shown to be necessary, but not sufficient. However, it can easily be seen that the principle of virtual work also implies the equilibrium equations (5.2.3) and (5.2.8), and as the only necessary ones: since the displacement components \( \bar{u}_\alpha, w \) can vary independently in \( A \), and since the area integral in Equation (5.2.8) must vanish, it follows that the coefficients of \( \delta \bar{u}_\alpha \) and \( \delta w \) must vanish. Equation (5.2.5) may then be used as the definition of \( Q_\alpha \).

Before introducing constitutive equations, it must be noted that although the displacement model is one in which \( \varepsilon_{33} = 0 \), this constraint is not realistic. Actually, it is the stress \( \sigma_{33} \) which is very nearly zero, or at least, its maximum value is very small in comparison to those of the stresses \( \sigma_{\alpha\beta} \) \((\alpha, \beta = 1, 2)\). Similarly, the shear stresses \( \sigma_{3\alpha} \), though important in the equilibrium equations, are generally of small magnitude. Consequently, most points of the plate are nearly in a state of plane stress. The elastic behavior of isotropic plates should therefore be described by Equations (1.4.13), and plasticity by a plane-stress yield criterion.

### 5.2.2. Elastic Plates

**Elastic Relations**

Equations (1.4.13),

\[
\sigma_{\alpha\beta} = \frac{E}{1 - \nu^2} [(1 - \nu)\varepsilon_{\alpha\beta} + \nu\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}],
\]
lead to

\[ N_{\alpha\beta} = \frac{Eh}{1 - \nu^2} [(1 - \nu) \bar{\varepsilon}_{\alpha\beta} + \nu \bar{\varepsilon}_{\gamma\gamma} \delta_{\alpha\beta}] \]

and

\[ M_{\alpha\beta} = D [(1 - \nu) \kappa_{\alpha\beta} + \nu \kappa_{\gamma\gamma} \delta_{\alpha\beta}], \]

where \( D = \frac{Eh^3}{12(1 - \nu^2)} \) is the plate bending modulus, and \( \kappa_{\alpha\beta} = w_{,\alpha\beta} \) is the curvature tensor.

It can be seen that the problem of the in-plane forces is identical with the plane-stress problem, with \( N_{\alpha\beta}, \sigma_{\alpha\beta}, \bar{u}_{\alpha} \) and \( \bar{\varepsilon}_{\alpha\beta} \) corresponding to \( \sigma_{\alpha\beta}, T_{\alpha}, u_{\alpha}, \) and \( \varepsilon_{\alpha\beta} \), respectively. For the flexure problem, the equilibrium equation, when combined with the moment-curvature and curvature-deflection relations, becomes

\[ \nabla^4 w = \frac{\bar{q}}{D}, \]

where \( \bar{q} = q - m_{\alpha,\alpha} \) is the effective transverse load per unit area. In what follows we shall assume, as is the case in most problems, that \( m_{\alpha} = 0 \) and therefore \( \bar{q} \) will be replaced by \( q \).

**Axisymmetrically Loaded Circular Plates**

Given a circular plate of radius \( a \), if the load \( q \) is a function (in polar coordinates) of \( r \) only and if the edge conditions are uniform, then the deflection \( w \) can likewise be assumed to be a function of \( r \) only, the only nonzero shear force is \( Q_r = Q \), and the only moments are \( M_r \) and \( M_\theta \). Equation (5.2.4) then reduces to

\[ \frac{1}{r} \frac{d}{dr} (r Q) + q = 0, \]

which can be integrated to yield

\[ Q = -\frac{1}{r} \int_0^r rq \, dr, \]

and Equation (5.2.5) becomes

\[ \frac{dM_r}{dr} + \frac{M_r - M_\theta}{r} + Q = 0. \]

The curvature tensor components are

\[ \kappa_r = \frac{d^2w}{dr^2}, \quad \kappa_\theta = \frac{1}{r} \frac{dw}{dr}, \]

and therefore the elastic relations take the form

\[ M_r = D \left( \frac{d^2w}{dr^2} + \nu \frac{1}{r} \frac{dw}{dr} \right), \quad M_\theta = D \left( \frac{1}{r} \frac{dw}{dr} + \nu \frac{d^2w}{dr^2} \right). \]
Substituting this result in the moment-shear equations results in

\[ D \left( \frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} \right) = -Q. \]

But the left-hand side is just

\[ D \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dw}{dr} \right) \right], \]

so that the differential equation can be solved by integration.

The simplest problem is the one where the load is uniform, that is, \( q = \) constant. Then \( Q = -qr/2 \), and the integration for \( w \) results in

\[ w(r) = \frac{qr^4}{64D} + Ar^2 + B \ln r + C, \]

where \( A, B, C \) are constants of integration. For the deflection to be finite at the center we must have \( B = 0 \), and it is convenient to set \( C = 0 \), that is, measure the deflection relative to the center rather than relative to the edge. The remaining constant, \( A \), is then determined from the edge condition.

**Clamped Edge.** Here the edge condition is \( w'(a) = 0 \), leading to \( A = -qa^2/32D \). The deflection of the edge relative to the center is thus \( qa^4/64D \), or equivalently, the center deflection relative to the edge is \( qa^4/64D \). The moments are

\[ M_r(r) = \frac{q}{16} [(3 + \nu)r^2 - (1 + \nu)a^2], \quad M_\theta(r) = \frac{q}{16} [(1 + 3\nu)r^2 - (1 + \nu)a^2]. \]

**Simply Supported Edge.** We have

\[ M_r(r) = \frac{(3 + \nu)qr^2}{16} + 2(1 + \nu)DA, \]

and therefore the condition \( M_r(a) = 0 \) leads to \( A = -(3 + \nu)qa^2/32(1 + \nu)D \). The deflection is therefore

\[ w(r) = \frac{q}{64(1 + \nu)D} [(1 + \nu)r^4 - 2(3 + \nu)a^2r^2], \]

the maximum deflection being \((5 + \nu)qa^4/(64(1 + \nu)D)\), or, with \( \nu = 0.3 \), about four times as large as for the clamped plate. The moments are

\[ M_r(r) = \frac{(3 + \nu)q(r^2 - a^2)}, \quad M_\theta(r) = \frac{q}{16} [(1 + 3\nu)r^2 - (3 + \nu)a^2]. \]

As a preliminary step to determining the deflection due to an arbitrary axisymmetric load \( q(r) \), we consider the case of a force \( F \) concentrated on a
circle of radius $b$. This may be viewed as the limit as $c \to b$ of the annular loading

$$q(r) = \begin{cases} 
0, & 0 < r < b, \\
\frac{F}{\pi(c^2 - b^2)}, & b < r < c, \\
0, & c < r < a.
\end{cases}$$

It follows that $rQ$ is constant for $r < b$ and for $r > c$ (with $Q = 0$ in the former region), and that

$$rQ|_b^c = -\int_b^c q(r) r \, dr = -\frac{F}{2\pi}.$$ 

Consequently, in the limit as $c \to b$ we have

$$\frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dw}{dr} \right) \right] = \begin{cases} 
0, & r < b, \\
\frac{F}{2\pi D}, & r > b.
\end{cases}$$ 

Denoting the deflections in $r < b$ and $r > b$ by $w_1$ and $w_2$, respectively, and setting $w_1(0) = 0$ for convenience, we have

$$w_1(r) = A_1 r^2,$$

$$w_2(r) = A_2 r^2 + B_2 b^2 \ln \frac{r}{b} + C_2 b^2 + \frac{F}{8\pi D} r^2 \ln \frac{r}{b}.$$ 

The continuity conditions $w_1(b) = w_2(b)$, $w_1'(b) = w_2'(b)$, and $w_1''(b) = w_2''(b)$ yield

$$A_2 = A_1 - \frac{F}{8\pi D}, \quad B_2 = C_2 = \frac{F}{8\pi D}.$$ 

Thus

$$w_2(r) = A_1 r^2 + \frac{F}{8\pi D} \left[ (r^2 + b^2) \ln \frac{r}{b} + b^2 - r^2 \right],$$

$$w'_2(r) = 2A_1 r + \frac{F}{8\pi D} \left( 2r \ln \frac{r}{b} + \frac{b^2}{r} - r \right),$$

and in $r > b$,

$$M_r(r) = 2(1 + \nu) D A_1 + \frac{F}{8\pi} \left[ 2(1 + \nu) \ln \frac{r}{b} + (1 - \nu) \left( 1 - \frac{b^2}{r^2} \right) \right].$$ 

If the edge $r = a$ is clamped, then $w_2'(a) = 0$ and therefore

$$A_1 = -\frac{F}{8\pi D} \left[ \ln \frac{a}{b} - \frac{1}{2} \left( 1 - \frac{b^2}{a^2} \right) \right].$$
while if the edge is simply supported, then \( M_r(a) = 0 \) and
\[
A_1 = -\frac{F}{8\pi D} \left[ \ln \frac{a}{b} + \frac{1 - \nu}{2(1 + \nu)} \left( 1 - \frac{b^2}{a^2} \right) \right].
\]
These results may be combined by writing
\[
A_1 = -\frac{F}{8\pi D} \left[ \ln \frac{a}{b} + \lambda \left( 1 - \frac{b^2}{a^2} \right) \right],
\]
where \( \lambda \) equals \(-\frac{1}{2}\) for the clamped plate and \((1 - \nu) / 2(1 + \nu)\) for the simply supported plate. Since we have set \( w(0) = 0 \), the center deflection relative to the edge is
\[
-w(a) = \frac{F}{8\pi D} \left[ b^2 \ln \frac{a}{b} - (1 + \lambda)(a^2 - b^2) \right].
\]
Let the solution derived above be written as
\[
w(r) = -\frac{F}{8\pi D} g(r, b; \lambda),
\]
where, by definition,
\[
g(r, \rho; \lambda) = \begin{cases} 
\left[ \ln \frac{a}{\rho} + \lambda \left( 1 - \frac{\rho^2}{a^2} \right) \right] r^2, & r < \rho, \\
r^2 \ln \frac{a}{r} - \rho^2 \ln \frac{r}{\rho} + (1 + \lambda) r^2 - \left( 1 + \lambda \frac{\rho^2}{a^2} \right) \rho^2, & r > \rho.
\end{cases}
\]
Now consider an arbitrary load distribution \( q(r) \). The total load contained in the infinitesimal annulus \( \rho < r < \rho + d\rho \) is \( 2\pi q(\rho) \rho d\rho \), and consequently the deflection due to this load alone is \(- (1/4D) q(\rho) g(r, \rho; \lambda) \rho d\rho \). The deflection due to the entire load is obtained by superposition:
\[
w(r) = -\frac{1}{4D} \int_0^a q(\rho) g(r, \rho; \lambda) \rho d\rho,
\]
and for the center deflection, in particular, we have
\[
-w(a) = \frac{1}{4D} \int_0^a q(\rho) \left[ (1 + \lambda)(a^2 - \rho^2) - \rho^2 \ln \frac{a}{\rho} \right] \rho d\rho.
\]
Consider, for example, a downward load \( F \) uniformly distributed over an inner circle of radius \( b \); then
\[
w(a) = \frac{F}{4\pi b^2 D} \int_0^b \left[ (1 + \lambda)(a^2 - \rho^2) - \rho^2 \ln \frac{a}{\rho} \right] \rho d\rho
\]
\[
= \frac{F}{4\pi D} \left[ (1 + \lambda) \left( \frac{a^2}{2} - \frac{b^2}{4} \right) - \frac{b^2}{4} \ln \frac{a}{b} + \frac{b^2}{16} \right].
\]
When $b = a$, we obtain

$$w(a) = [4(1 + \lambda) - 1] \frac{Fa^2}{64\pi D},$$

from which we obtain the previous results for the clamped and simply supported cases (with $q = -F/\pi a^2$) by inserting the appropriate values of $\lambda$.

With $b = 0$ we obtain the solution for a load concentrated at the center. The deflection at any $r$ is obtained by evaluating $w(r)$ by superposition, with $r > b$, and then taking the limit as $b \to 0$:

$$w(r) = \lim_{b \to 0} \frac{1}{4D} \cdot \frac{F}{\pi b^2} \int_0^b \left[ r^2 \ln \frac{a}{r} - \rho^2 \ln \frac{r}{\rho} + (1 + \lambda)r^2 - \left(1 + \frac{\lambda r^2}{a^2}\right)\rho^2 \right] \rho \, d\rho$$

$$= \frac{F}{8\pi D} \left[ r^2 \ln \frac{a}{r} + (1 + \lambda)r^2 \right].$$

The maximum deflection is

$$w(a) = \frac{Fa^2}{16\pi D} \left\{ \begin{array}{ll} 1, & \text{clamped,} \\ \frac{3 + \nu}{1 + \nu}, & \text{simply supported.} \end{array} \right.$$  

The bending moments are

$$M_r = \frac{F}{8\pi} \left[ 2(1 + \nu) \ln \frac{a}{r} - (1 - \nu) + 2(1 + \nu)\lambda \right],$$

$$M_\theta = \frac{F}{8\pi} \left[ 2(1 + \nu) \ln \frac{a}{r} + (1 - \nu) + 2(1 + \nu)\lambda \right].$$

The logarithmic singularity at the center indicates that the assumptions of elementary plate theory do not hold near the point of application of a concentrated load. For more details, see Timoshenko and Woinowsky-Krieger [1959], Section 5.1.

### 5.2.3. Yielding of Plates

#### Plate Yield Criteria

A plate will be said to yield in bending at a point $(x_1, x_2)$ if the stress tensor there obeys the yield criterion at every $x_3$ except $x_3 = 0$; points in the middle plane are regarded as remnants of the elastic core. If the stresses $\sigma_{i3}$ are assumed negligible in magnitude next to the $\sigma_{\alpha\beta}$ (this does not mean that they can be neglected in the equilibrium equations, because derivatives occur there), then we may apply a plane-stress yield criterion, say

$$f \left( \frac{\sigma_{\alpha\beta}}{\sigma_y} \right) = 0,$$
in every plane \( x_3 \neq 0 \). Equilibrium is satisfied if

\[
\sigma_{\alpha\beta} = -\frac{4}{h^2} M_{\alpha\beta} \text{sgn} x_3. \tag{5.2.9}
\]

If the ultimate moment is defined as \( M_U = \sigma_y h^2 / 4 \), then the plate yield criterion is given by

\[
f(m_{\alpha\beta}) = 0,
\]

where \( m_{\alpha\beta} = M_{\alpha\beta} / M_U \); the Mises and Tresca criteria become, respectively,

\[
m_{11}^2 - m_{11} m_{22} + m_{22}^2 + 3m_{12}^2 = 1 \quad \text{(Mises)},
\]

\[
\max(|m_1|, |m_2|, |m_1 - m_2|) = 1 \quad \text{(Tresca)}.
\]

A yield criterion that approximates the behavior of doubly reinforced concrete slabs is the Johansen criterion,

\[
\max(|m_1|, |m_2|) = 1.
\]

Problems in contained plastic bending of plates may be studied by means of numerical methods analogous to those of Section 4.5.\(^\text{1}\) The plastic collapse of perfectly plastic plates may in general be investigated by means of limit analysis, as is done in Section 6.4. If such plates are, however, circular and axisymmetrically loaded and supported, then exact solutions may be obtained for the collapse state.

**Fully Plastic Axisymmetrically Loaded Circular Plates**

If the loading and support are axisymmetric, then the only nonvanishing moments are \( M_r \) and \( M_\theta \), and the equilibrium equation is

\[
(r M_r)' - M_\theta = \int_0^r qr \, dr. \tag{5.2.10}
\]

This equation and the yield condition constitute two equations for \( M_r \) and \( M_\theta \). Equivalently, if the yield condition is solved for \( M_\theta \) in terms of \( M_r \) and the resulting expression for \( M_\theta \) is substituted in (5.2.10), the result is a nonlinear first-order differential equation for \( M_r(r) \). At the center of the plate, \( M_r = M_\theta \) and consequently, if the curvature there is positive (concave upward), \( M_r(0) = M_U \) constitutes an initial condition with which the differential equation may be solved. In addition, a boundary condition at the edge \( r = a \) must be satisfied; this yields the ultimate load. Let us recall that for a simply supported plate, the edge conditions are \( w = M_r = 0 \); thus \( M_r = 0 \) is a boundary condition with which the differential equation may be solved. For a clamped plate, the edge must form a hinge circle.

\(^1\)For an introduction to finite-element methods for plates, see Zienkiewicz [1977], Chapter 10.
that is, a locus of slope discontinuity (a special case of the hinge curve discussed in Section 6.2). As we shall see, the edge condition there becomes \( M_r(a) = -M_U \) or \( M_r(a) = -2M_U/\sqrt{3} \) for the Tresca or Mises material, respectively.

Figure 5.2.2 shows the Mises and Tresca yield criteria for axisymmetrically loaded circular plates. It follows from the preceding discussion that the center of the plate is in the moment state corresponding to point \( B \), and that a simply supported edge corresponds to point \( C \). A simply supported plate may thus be assumed to be entirely in the regime \( BC \). For the Tresca material, this means that \( M_\theta = M_U \) everywhere, and the problem to be solved is therefore linear.

\[ \text{Figure 5.2.2. Mises and Tresca yield criteria for axisymmetrically loaded circular plates} \]

**Solution for Tresca Plate**

Let us consider, for example, a downward load \( F \) uniformly distributed over a circle of radius \( b \), the plate being unloaded outside this circle. The equilibrium equation is then

\[
(rM_r)' - M_U = \begin{cases} 
-\frac{Fr^2}{2\pi b^2}, & r < b, \\
-\frac{F}{2\pi}, & r > b.
\end{cases}
\]

The solution for \( r < b \) satisfying the condition at \( r = 0 \) is

\[ M_r = M_U - \frac{Fr^2}{6\pi b^2}, \]

while the solution for \( r > b \) satisfying the condition at \( r = a \) is

\[ M_r = \left( \frac{F}{2\pi} - M_U \right) \left( \frac{a}{r} - 1 \right). \]
Continuity at \( r = b \) requires that

\[
F = 2\pi \frac{M_U}{1 - 2b/3a}.
\]

This result includes the extreme cases \( F = 6\pi M_U \) for the uniformly loaded plate \((b = a)\) and \( F = 2\pi M_U \) for a plate with concentrated load. This last case could not have been treated directly because the moments would have to go to infinity at the center—a condition incompatible with plasticity.

On segment \( BC \) of the Tresca hexagon, the flow rule yields \( \dot{\kappa}_r/\dot{\kappa}_\theta = 0 \), that is, \( \dot{w}''/(\dot{w}'/r) = 0 \). Consequently the velocity field obeying the edge condition must be \( \dot{w}(r) = (1 - r/a)v_0 \), where \( v_0 \) is the center velocity (i.e., the plate deforms in the shape of a cone). It is shown in Chapter 6 that the upper-bound load obtained with this velocity field equals the one obtained here, as indeed it must, since the solution is complete.

In order to study the clamped plate, we must know the velocity fields associated with the other sides of the Tresca hexagon. On \( CD \) we have \( \dot{\kappa}_r + \dot{\kappa}_\theta = 0 \), that is, \( \dot{w}'' + \dot{w}'/r = 0 \), a differential equation whose solution is \( \dot{w} = C + D \ln r \). Neither this velocity field nor the preceding (conical) one can possibly meet a condition of zero slope at a clamped edge, and this is why a hinge circle is necessary there. Finally, on \( AB \) and \( DE \) the flow rule gives \( \dot{\kappa}_\theta/\dot{\kappa}_r = 0 \). If this is interpreted as \( \dot{\kappa}_\theta = \dot{w}'/r = 0 \), then the velocity field is \( \dot{w} = \text{constant} \) (i.e., rigid-body motion). Alternatively, we may interpret it as \( \dot{\kappa}_r = \text{inf} \); this would be the state at a hinge circle, and from this follows the clamped-edge condition \( M_r = -M_U \). On the Mises ellipse, the only relevant point where \( \dot{\kappa}_\theta/\dot{\kappa}_r = 0 \) is \( D' \), where \( M_r = -2M_U/\sqrt{3} \).

It follows from these considerations that a clamped plate must be in regime \( BC \) near the center and in \( CD \) near the edge; point \( C \) gives the state at \( r = c \) for some \( c \) such that \( 0 < c < a \). With the same loading as assumed for the simply supported plate above, we have to solve the problem both for \( c > b \) and \( c < b \).

**Case 1: \( c > b \).** The equilibrium equation is

\[
(rM_r)' = M_U - \frac{Fr^2}{2\pi b^2}, \quad 0 < r < b,
\]

\[
(rM_r)' = M_U - \frac{F}{2\pi}, \quad b < r < c,
\]

\[
rM_r' = M_U - \frac{F}{2\pi}, \quad c < r < a.
\]

Let \( m = M_r/M_U \) and \( p = F/2\pi M_U \); then the solution satisfying the cond-
tions at \( r = 0 \) and \( r = c \) is

\[
m(r) = 1 - \frac{pr^2}{3b^2}, \quad 0 < r < b, \]
\[
m(r) = (p - 1) \left( \frac{c}{r} - 1 \right), \quad b < r < c, \]
\[
m(r) = -(p - 1) \ln \frac{r}{c}, \quad c < r < a.
\]

The additional conditions to be met are continuity at \( r = b \) and \( m = -1 \) at \( r = a \), producing the two equations

\[
1 - \frac{1}{3} p = (p - 1) \left( \frac{c}{b} - 1 \right),
\]
\[
(p - 1) \ln \frac{a}{c} = 1.
\]

The assumption \( c > b \) requires \( 1 < p < 3 \), and the limiting case \( c = b \), \( p = 3 \) corresponds to \( b/a = e^{-1/2} \). Consequently, the present case represents the range \( 0 < b/a < e^{-1/2} \). Eliminating \( c \) between the two equations, we obtain

\[
\frac{b}{a} = 3(p - 1) \frac{e^{-\frac{1}{2} - p}}{2p}, \quad p < 3.
\]

For example, \( p = 2 \) corresponds to \( b/a = 0.276 \), \( p = 1.5 \) to \( b/a = 0.068 \), and \( p = 1.286 \) to \( b/a = 0.01 \). The ultimate concentrated load is \( 2\pi M_U \) \((p = 1, \ b = 0)\), the same as for the simply supported plate. This limit is approached, however, only at extremely small values of \( b/a \).

**Case 2**: \( b > c \). The equilibrium equation is

\[
(r M_r)' = M_U - \frac{F r^2}{2 \pi b^2}, \quad 0 < r < c,
\]
\[
r M_r' = M_U - \frac{F r^2}{2 \pi b^2}, \quad c < r < b,
\]
\[
r M_r' = M_U - \frac{F}{2\pi}, \quad b < r < a.
\]

An analysis similar to that in Case 1 leads to

\[
\frac{b}{a} = e^{-\frac{5 - p + \ln(p/3)}{2(p - 1)}}, \quad p > 3.
\]

The extreme case \( b = a \) (uniformly loaded plate) corresponds to \( p = 5.63 \), an increase of 88% over the ultimate load of the simply supported case.

Much of the preceding theory is due to H. G. Hopkins and various collaborators. For more details and other solutions, see, for example, Hopkins and Prager [1953], Hopkins and Wang [1954], and Drucker and Hopkins [1955].
For plates without circular symmetry, complete solutions are not available. Estimates of collapse loads will be found with the help of the upper-bound theorem of limit analysis — and exceptionally of the lower-bound theorem — in 6.4.1.

The effect of large deflections on the collapse load of a simply supported circular plate was studied by Onat and Haythornthwaite [1956]; the analysis is based on the deformation of the plate into a conical shell (see also Hodge [1959], Section 11-7). Approximate methods for determining collapse loads of shells are studied in 6.4.3.

**Exercises: Section 5.2**

1. Derive the plate equilibrium equations (5.2.6) by means of virtual work. Discuss the significance of the shear forces $Q_\alpha$ in this formulation.

2. A uniformly loaded circular plate made of an elastic–perfectly plastic material with yield stress $\sigma_Y$ is assumed to obey the Tresca criterion. If the total load is $F$, determine the value $F_E$ at which yielding begins when the plate is (a) simply supported and (b) clamped. Compare with the corresponding ultimate loads.

3. Determine the velocity field in a fully plastic clamped circular Tresca plate of radius $a$ carrying a load $F$ that is uniformly distributed over an inner circle of radius $b$. Use the upper-bound theorem to determine the relation between the load and the ratio $b/a$. Compare with the results in the text.

4. A simply supported circular Tresca plate of radius $a$ carries a load $F$ that is uniformly distributed over the perimeter of the circle $r = b$. Find the value of $F$ and the moment distribution when the plate is fully plastic.

5. A simply supported circular Tresca plate of radius $a$ carries a load $F$ that is uniformly distributed over the entire plate. The ultimate moment is $M_U$ inside the circle $b = a/2$ and $\frac{1}{4}M_U$ outside this circle. Find the value of $F$ and the moment distribution when the plate is fully plastic.

**Section 5.3 Plastic Buckling**

While Sections 5.1 and 5.2 dealt with the plastic collapse of bodies whose material behavior may be idealized as perfectly plastic, the present section
is devoted to the study of the buckling collapse of bodies made of work-hardening material. The elementary theory of elastic column buckling is part of the knowledge of all students of mechanics. It is worthwhile, however, to begin this section, in 5.3.1, with a general introduction to stability theory and to use the buckling of bars as an illustration of the theory. In 5.3.2 we discuss theories of the modulus that must be used in the determination of critical loads. Finally, in 5.3.3 the plastic buckling of plates and shells is studied.

5.3.1. Introduction to Stability Theory

Elastic Stability

In 1.4.3 it was shown that an elastic body under conservative loads is in equilibrium if and only if the total potential energy $\Pi$ is stationary with respect to virtual displacements, a condition expressed by Equation (1.4.18):

$$\delta \Pi = 0.$$ 

It was further stated without proof that the equilibrium is stable only if $\Pi$ is a minimum. An intuitive, though not strictly rigorous proof can be based on the following observation: if the configuration of the body is to change slightly from the one at equilibrium, and if the potential energy at the altered configuration is greater than at equilibrium, then additional work must be done on the body in order to effect the change, and therefore the change cannot take place spontaneously. The proof can be easily made rigorous for discrete systems (those with a finite number of degrees of freedom), and the result is known as the Lagrange–Dirichlet theorem. The proof for continua runs into technical difficulties, but these will be ignored here, and the result will be accepted.

Mathematically, the condition that $\Pi$ is a minimum at equilibrium can be expressed as follows: let $\Pi$ denote the potential energy evaluated at the displacement field $\mathbf{u}$, and $\Pi + \Delta \Pi$ the potential energy evaluated at the varied displacement field $\mathbf{u} + \delta \mathbf{u}$. Assuming the dependence of $\Pi$ on $\mathbf{u}$ to be smooth, we can write

$$\Delta \Pi = \delta \Pi + \frac{1}{2} \delta^2 \Pi + \ldots,$$

where $\delta \Pi$ (the first variation defined in 1.4.3) is linear in $\delta \mathbf{u}$ (and/or in its derivatives, and therefore also in $\delta \varepsilon$), $\delta^2 \Pi$ is quadratic, and so on. We may limit ourselves to virtual displacements that are small enough so that terms beyond the quadratic can be neglected. Since $\delta \Pi$ vanishes if $\Pi$ is stationary at $\mathbf{u}$, clearly $\Pi$ is a minimum only if $\delta^2 \Pi$ is nonnegative for all $\delta \mathbf{u}$ that are compatible with the constraints (i.e., for all virtual displacements).
We may therefore say that the equilibrium is stable only if 
\[ \delta^2 \Pi > 0 \]
for all virtual displacements. The criterion for the onset of instability, known as the **Trefftz criterion**, is thus

\[ \delta^2 \Pi = 0. \]

Since \( \Pi = \Pi_{int} + \Pi_{ext} \), we have \( \delta^2 \Pi = \delta^2 \Pi_{int} + \delta^2 \Pi_{ext} \). An equivalent statement of the Trefftz criterion is therefore

\[ \delta^2 \Pi_{int} + \delta^2 \Pi_{ext} = 0. \]

In a linearly elastic body,

\[ \Pi_{int} = \frac{1}{2} \int_R C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \, dV, \]

and therefore

\[ \Delta \Pi_{int} = \frac{1}{2} \int_R C_{ijkl} [(\varepsilon_{ij} + \delta \varepsilon_{ij})(\varepsilon_{kl} + \delta \varepsilon_{kl}) - \varepsilon_{ij} \varepsilon_{kl}] \, dV \]

\[ = \int_R C_{ijkl} \varepsilon_{ij} \delta \varepsilon_{kl} \, dV + \frac{1}{2} \int_R C_{ijkl} \delta \varepsilon_{ij} \delta \varepsilon_{kl} \, dV. \]

As a result of the definitions of the first and second variations, the first integral in the last expression is \( \delta \Pi_{int} \), and the second integral is \( \delta^2 \Pi_{int} \), that is,

\[ \delta^2 \Pi_{int} = \int_R C_{ijkl} \delta \varepsilon_{ij} \delta \varepsilon_{kl} \, dV. \]  \hspace{1cm} (5.3.1)

**Generalization to Quasi-Elastic Materials**

A general theory of stability in elastic–plastic solids capable of large deformations was formulated by Hill [1958]. Here, we shall limit our consideration to infinitesimal deformations. We can then immediately generalize the preceding result for linearly elastic bodies to nonlinearly elastic ones if we interpret \( C \) as the **tangent modulus** tensor defined by

\[ C_{ijkl} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}. \]

Since \( \delta \sigma_{ij} = C_{ijkl} \delta \varepsilon_{kl} \), the integrand of (5.3.1) can be rewritten as \( \delta \sigma_{ij} \delta \varepsilon_{ij} \).

Since it is a given equilibrium state that is examined as to its stability, the past history of the body is irrelevant to this examination, and therefore the result can be further extended to materials that are **quasi-elastic** in the sense that, at a given state, a small stress variation \( \delta \sigma \) can be uniquely associated with a small strain variation \( \delta \varepsilon \). As we have seen, rate-independent plastic materials have this property. We may therefore state, as a generalization to quasi-elastic bodies of the Trefftz criterion, the **energy criterion** for the stability of such bodies subject to conservative loads:

\[ \int_R \delta \sigma_{ij} \delta \varepsilon_{ij} \, dV + \delta^2 \Pi_{ext} = 0. \]  \hspace{1cm} (5.3.2)
Furthermore, the first term on the left-hand side will be written as $\delta^2 \Pi_{\text{int}}$ without thereby implying the existence of an internal potential energy $\Pi_{\text{int}}$. In other words, we shall define

$$\delta^2 \Pi_{\text{int}} = \int_R \delta \sigma_{ij} \delta \varepsilon_{ij} dV.$$ 

In the absence of internal constraints, the quantity $\delta \sigma_{ij} \delta \varepsilon_{ij}$ is positive for all nonvanishing $\delta \varepsilon$ if the material is linearly elastic. It is so likewise for rate-independent plastic materials that are stable in the sense of Drucker (see 3.2.1), that is, work-hardening and obeying an associated flow rule. For bodies made of such materials, then, instability can occur only if it is possible for $\delta^2 \Pi_{\text{ext}}$ to be negative.

If the loads $f$ and $t^a$ are independent of displacement, the external potential energy is

$$\Pi_{\text{ext}} = - \int_R f_i u_i dV - \int_{\partial R} t^a_i u_i dS,$$ 

as given in 1.4.3. We see, then, that $\Delta \Pi_{\text{ext}} = \delta \Pi_{\text{ext}}$ (i.e. $\delta^2 \Pi_{\text{ext}} = 0$) unless the virtual displacement significantly alters the region occupied by the body, thereby introducing terms that are quadratic in $\delta u$ into $\Delta \Pi$.

**Buckling of Quasi-Elastic Bars**

The classic example of this occurrence is the buckling of a bar (column, strut) under a compressive axial load $P$ acting through the neutral axis. If the bar is initially straight, with length $L$, and then bends, with the deflection of the neutral axis given by $v(x)$ ($0 < x < L$), then the chord spanned by the neutral axis becomes $\int_0^L \sqrt{1 - v'^2} dx$, so that the work done by the load $P$ is

$$P \left( L - \int_0^L \sqrt{1 - v'^2} dx \right) = \frac{1}{2} P \int_0^L v'^2 dx.$$ 

If the only other load is a distributed transverse load $q$, then the external potential energy is

$$\Pi_{\text{ext}} = - \int_0^L q v dx - \frac{1}{2} P \int_0^L v'^2 dx,$$ 

and the first and second variations are, respectively,

$$\delta \Pi_e = - \int_0^L q \delta v dx - P \int_0^L v' \delta v' dx$$

$$= \int_0^L (P v'' - q) \delta v dx - P v' \delta v' \bigg|_0^L$$

(the last form having been obtained by integration by parts), and

$$\delta^2 \Pi_{\text{ext}} = - P \int_0^L (\delta v')^2 dx.$$
If the bar is subject to a distributed axial compressive load of intensity \( p \) per unit length in addition to end loads \( P_0 \) and \( P_L \) at \( x = 0 \) and \( x = L \), respectively, then the external potential energy can be written as

\[
\Pi_{\text{ext}} = - \int_0^L qv \, dx - \int_0^L ps \, dx + P_0 s(0) - P_L s(L),
\]

where \( s(x) \) is the shortening of the bar due to bending at point \( x \), given by

\[
s(x) = s(0) + \frac{1}{2} \int_0^x v'^2 \, dx.
\]

If \( P(x) \) now denotes the internal axial force at \( x \) (positive in compression), then equilibrium requires

\[
P' + p = 0,
\]

and therefore

\[
- \int_0^L ps \, dx = \int_0^L P' s \, dx = P s|_0^L + \frac{1}{2} \int_0^L P v'^2 \, dx.
\]

Equation (5.3.3) for \( \Pi_{\text{ext}} \) needs to be changed only by placing \( P \) under the integral sign, and

\[
\delta^2 \Pi_{\text{ext}} = - \int_0^L P(\delta v')^2 \, dx.
\]

In accordance with elementary beam theory, the state of stress at each point will be approximated as uniaxial, so that \( \delta \sigma_{ij} \delta \varepsilon_{ij} = \delta \sigma_x \delta \varepsilon_x \). Furthermore, \( \delta \varepsilon_x = -y \delta v'' \), so that

\[
\delta^2 \Pi_{\text{int}} = \int_0^L \left[ \int_A (-y) \delta \sigma_x \, dA \right] \delta v'' \, dx = \int_0^L \delta M \delta v'' \, dx,
\]

where \( M \) is the bending moment. The energy criterion for bars therefore takes the form

\[
\int_0^L \delta M \delta v'' \, dx - \int_0^L P(\delta v')^2 \, dx = 0. \tag{5.3.4}
\]

Integration by parts of the first integral leads to

\[
\int_0^L (\delta M' + P \delta v') \delta v' \, dx = \delta M \delta v'|_0^L.
\]

Since the end conditions on beams are usually such that either the rotation \( v' \) or the bending moment \( M \) cannot be varied, the right-hand side of this equation vanishes.

Finally, we assume an effective modulus \( \bar{E} \) such that \( \delta M = \bar{E} I \delta v'' \); in a linearly elastic material, of course, this is just the Young’s modulus \( E \). In nonlinear materials \( \bar{E} \) may be assumed to be determined by the average
stress in the bar, $\sigma = P/A$. Theories of the effective modulus are discussed in 5.3.2.

In problems in which constraints on the deflection $v$ itself can be disregarded, the virtual rotation $\delta v'$ may be taken as the unknown variable. Writing this variable as $\theta$, we obtain, upon observing that the rotation is not constrained in the interior of the bar, the differential equation

$$\left( \bar{E}I \theta' \right)' + P \theta = 0. \quad (5.3.5)$$

More generally, the first integral in (5.3.4) must be integrated by parts twice, so that the energy criterion becomes

$$\int_0^L \left( \delta M' + P \delta v' \right) \delta v \, dx = \left. \left( \delta M' + P \delta v' \right) \delta v \right|_0^L. \quad (5.3.6)$$

The quantity in parentheses may be interpreted as the virtual shear force, and therefore either it or the virtual displacement can be expected to vanish at each end. The differential equation expressing the energy criterion is therefore

$$\left( \bar{E}I \theta' \right)'' + (P \theta)' = 0. \quad (5.3.7)$$

**End-Loaded Prismatic Bars**

If the bar is prismatic then $A$ and $I$ are constant. If, moreover, it is subject to an axial end load only, then $P$ is constant. In such a bar, then, the average stress $P/A$ and therefore, by hypothesis, the effective modulus are constant. Equation (5.3.7) thus becomes

$$\bar{E}I \theta'' + P \theta' = 0, \quad (5.3.8)$$

and can be immediately solved as

$$\theta = B \cos \lambda x + C \sin \lambda x + D,$$

where $B$, $C$, and $D$ are constants, and $\lambda$ is defined by

$$\lambda^2 = \frac{P}{\bar{E}I}.$$

If the bar is pinned at both ends, then $\theta'(0) = 0$ and $\theta'(L) = 0$. The first condition requires $C = 0$, and the second

$$\lambda B \sin \lambda L = 0.$$

For a nontrivial solution, $^1$ $\lambda$ and $B$ must both be different from zero, and instability occurs only if

$$\sin \lambda L = 0,$$

$^1$Note that $D$ is irrelevant since it represents a rigid rotation.
that is, if \( \lambda = n\pi/L \), where \( n \) is a positive integer. The \textit{fundamental mode} of buckling corresponds to \( n = 1 \), that is, \( \lambda = \pi/L \), and the load producing it (the \textit{critical load}) obeys
\[
P = \frac{\pi^2}{E} \frac{I}{L^2}.
\]
If the bar is elastic, \( \bar{E} = E \), and
\[
P = \pi^2 \frac{EI}{L^2} \equiv P_E,
\]
where \( P_E \) is known as the \textit{Euler load}.

For other end conditions, the equation governing the critical load can be written as
\[
P = \frac{\pi^2}{\bar{E}} \frac{I}{L^2} = P_E,
\]
where \( L_e \) is known as the \textit{effective length} of the bar, given by \( L_e = \pi/\lambda_1 \), \( \lambda_1 \) being the smallest nonzero value of \( \lambda \). For a cantilever column, for example, the characteristic equation is \( \cos \lambda L = 0 \), leading to a fundamental mode described by \( \lambda_1 = \pi/2L \), and hence \( L_e = 2L \). For a bar that is clamped at \( x = 0 \) and pinned at \( x = L \), the end conditions are \( \theta(0) = \theta'(L) = 0 \), in addition to the constraint \( \int_0^L \theta \, dx = 0 \), describing zero deflection of the pinned end relative to the clamped end. The three conditions lead to the characteristic equation \( \lambda L = \tan \lambda L \), whose lowest root is \( \lambda_1 L = 4.4934 \). Hence \( L_e/L = \pi/4.4934 = 0.699 \).

The solution of Equation (5.3.9) for the critical load for inelastic bars will be postponed until after the discussion of the effective modulus.

\subsection*{5.3.2. Theories of the Effective Modulus}

\textit{Tangent-Modulus (Engesser–Shanley) Theory}

The first analysis of inelastic column buckling is due to Engesser [1889], who based the calculation of the critical load on the incremental relation \( \delta \sigma_x = E_t \delta \varepsilon_x \), where \( E_t \) is the tangent modulus, defined as the slope \( d\sigma/d\varepsilon \) of the uniaxial compression curve at the current value of \( \sigma = P/A \). The essential assumption is that, while a bending moment requires a nonuniform stress distribution, the deviation of the stress from the average is, at least initially, sufficiently small so that the stress-strain curve can be locally approximated by straight line with slope \( E_t \). Consequently, the effective modulus \( \bar{E} \) is just \( E_t \). Since this is a function of \( \sigma \), it is convenient to rewrite Equation (5.3.9) in terms of \( \sigma \) rather than \( P \), and to designate the solution as the \textit{critical stress} \( \sigma_{cr} = P_{cr}/A \). It is conventional to define the “radius of gyration” \( r \equiv \sqrt{I/A} \), and to designate \( L_e/r \) as the \textit{slenderness
ratio. Equation (5.3.9) now reads

$$\frac{\sigma}{E_t(\sigma)} = \frac{\pi^2}{(L_e/r)^2}.$$  \hspace{1cm} (5.3.10)

A plot of the solution of this equation, $\sigma_{cr}$ against $L_e/r$, is often called a column curve. When $\sigma_{cr}$ is given explicitly as a function of $L_e/r$, the relation is called a column formula.

If the material has a definite elastic-limit stress $\sigma_E$ such that $E_t(\sigma) = E$ for $\sigma < \sigma_E$, then the slenderness ratio at which $\sigma = \sigma_E$ is called the critical slenderness ratio, defined by

$$\left(\frac{L}{r}\right)_{cr} = \pi \sqrt{\frac{E}{\sigma_E}}.$$  \hspace{1cm} (5.3.11)

The critical slenderness ratio is clearly a material property (for mild steel it is about 90), and for supercritically slender bars the critical load is given by the elastic solution. The portion of the column curve for $L_e/r > (L/r)_{cr}$ is thus the Euler hyperbola given by

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2}.$$  \hspace{1cm} (5.3.12)

If the material is perfectly plastic, so that $\sigma_Y = \sigma_E$, then a stress greater than $\sigma_E$ is not possible, and therefore $\sigma_{cr} = \sigma_E$ for $L_e/r \leq (L/r)_{cr}$. For materials without a definite elastic limit, the column curve approaches the Euler hyperbola asymptotically as $L_e/r \to \infty$. For a material whose uniaxial stress-strain relation is described by the Ramberg–Osgood equation (2.1.2), the tangent modulus is easily obtained as

$$E_t = \frac{E}{1 + \alpha m (\sigma/\sigma_R)^m},$$  \hspace{1cm} (5.3.13)

and the column curve is obtained from

$$\frac{\sigma}{\sigma_R} + \alpha m \left(\frac{\sigma}{\sigma_R}\right)^m = \frac{\pi^2 E/\sigma_R}{(L_e/r)^2}.$$  \hspace{1cm} (5.3.14)

Since such a material hardens indefinitely, no cutoff stress exists for short bars.

A formula describing a stress-strain relation with no definite elastic limit that approaches perfect plasticity asymptotically, with an ultimate stress $\sigma_\infty$, was proposed by Prager [1942] in the form

$$\sigma = \sigma_\infty \tanh \frac{E \varepsilon}{\sigma_\infty}.$$  \hspace{1cm} (5.3.15)
The asymptote is approached quite fast: when the total strain equals twice the elastic strain \((\varepsilon = 2\sigma/E)\), the stress is already given by \(\sigma = 0.9575\sigma_\infty\). The tangent modulus can be readily obtained as

\[
E_t = E \left[ 1 - \left( \frac{\sigma}{\sigma_\infty} \right)^2 \right].
\]

The preceding formula can be easily generalized to

\[
E_t = E \left[ 1 - \left( \frac{\sigma}{\sigma_\infty} \right)^n \right],
\]

where \(n = 2\) corresponds to the Prager formula. The case \(n = 1\) describes an exponential stress-strain curve, \(\sigma = \sigma_\infty \left[ 1 - \exp\left( -E\varepsilon/\sigma_\infty \right) \right]\). The greater the value of \(n\), the more rapid the approach to perfect plasticity. The column curve for the generalized Prager formula may be plotted from

\[
\frac{\sigma/\sigma_\infty}{1 - (\sigma/\sigma_\infty)^n} = \frac{\pi^2 E/\sigma_\infty}{(L_c/r)^2}.
\]

Explicit column formulas can be obtained from this equation for \(n = 1\) and \(n = 2\).

**The Reduced-Modulus (Kármán) Theory**

Soon after the publication of Engesser’s theory, it was recognized by engineers, beginning with Considère [1891] and eventually including Engesser himself, that the tangent-modulus theory was in contradiction with the elastic–plastic behavior of metals; a formal theory was proposed by von Kármán [1910], based on the following reasoning.

When an initially straight bar under a compressive axial load begins to buckle, the fibers on the concave side undergo additional compression, but in those on the convex side the compressive strain, and hence the stress, is reduced. The stress change in the latter fibers is consequently elastic. Writing, for convenience, stress and strain as positive in compression, the incremental stress-strain relation is accordingly

\[
\delta \sigma = \begin{cases} 
E_t \delta \varepsilon, & \delta \varepsilon > 0, \\
E \delta \varepsilon, & \delta \varepsilon < 0,
\end{cases}
\]

where \(E_t\) is, as before, the elastic–plastic tangent modulus at the stress \(\sigma\). Let \(y = y_0\) give the location of the neutral fibers, with \(y > y_0\) and \(y < y_0\) being the areas of additional and reduced compression, respectively. Since \(\delta \varepsilon = (y - y_0) \delta \nu''\), the additional bending moment is

\[
\delta M = \int_A (y - y_0) \delta \sigma \, dA
\]
where
\[ E_r = \frac{1}{E} \left[ E \int_{y<y_0} (y-y_0)^2 \, dA + E_t \int_{y>y_0} (y-y_0)^2 \, dA \right] \] (5.3.13)
is the reduced modulus. The value of \( y_0 \) is determined, as usual, by the constancy of the axial load:
\[ 0 = \delta P = \int_A \delta \sigma \, dA, \]
or
\[ E \int_{y<y_0} (y-y_0) \, dA + E_t \int_{y>y_0} (y-y_0) \, dA = 0. \] (5.3.14)
Elimination of \( y_0 \) permits the expression of \( E_r \) as a function of \( \sigma \) and of bar geometry.

If the bar is rectangular, with width \( b \) and depth \( h \), then Equation (5.3.13), with the help of \( I = bh^3/12 \), gives
\[ E_r = \frac{4}{h^3} (E_t h_1^3 + Eh_2^3), \]
where
\[ h_1 = \frac{h}{2} - y_0, \quad h_2 = \frac{h}{2} + y_0. \]
Equation (5.3.14), with the factor \( b \) omitted, then becomes
\[ E \int_{-h/2}^{y_0} (y-y_0) \, dy + E_t \int_{y_0}^{h/2} (y-y_0) \, dy = \frac{1}{2} (E_t h_1^2 - Eh_2^2) = 0, \]
so that \( h_1/h_2 = \sqrt{E/E_t} \), and
\[ \frac{h_1}{h} = \frac{\sqrt{E}}{\sqrt{E} + \sqrt{E_t}}, \quad \frac{h_2}{h} = \frac{\sqrt{E_t}}{\sqrt{E} + \sqrt{E_t}}. \]
Finally, then,
\[ E_r = \frac{4EE_t}{(\sqrt{E} + \sqrt{E_t})^2}. \]

The reduced modulus is seen to be greater than the tangent modulus by a factor between 1 and 4, and therefore gives a correspondingly greater critical load.

Figure 5.3.1 shows column curves based on both the tangent-modulus and reduced-modulus theories, for the Ramberg–Osgood formula with \( \alpha = 0.1 \) and \( m = 10 \), and for the Prager formula.
Comparison of the Two Theories

The reduced-modulus theory dominated engineering practice for most of the first half of the twentieth century, since it was rigorously based on elastic–plastic theory. However, experiments in which bars are subjected to increasing axial loads until they buckle have consistently shown the results to be in much better agreement with the tangent-modulus theory. The first explanation of this discrepancy was given by Shanley [1947].

The reduced-modulus theory is based on the assumption that buckling occurs with no first-order change in the axial load, $\delta P = 0$. But, as Shanley wrote, “upon reaching the critical tangent-modulus load, there is nothing to prevent the column from bending simultaneously with increasing axial load.” It is thus possible for the neutral fibers to coincide initially with the extreme fibers on the concave side, all the other fibers undergoing additional compression. Any other possible location of the neutral fibers would correspond to a higher axial load, and therefore the tangent-modulus load $P_{tm}$ is a lower bound to the elastic–plastic buckling load.

The tangent-modulus load is thus not a load at which instability occurs, but bifurcation: when this load is exceeded the bar may be in one of several configurations — it may remain straight, or it may be bent in either direction.
(buckling in one plane only is assumed). If the bar buckles at $P_{tm}$, then, as the load is increased, the neutral axis moves inward, tending asymptotically to the position corresponding to the reduced-modulus load. The latter load is therefore an upper bound to the buckling load.

**Effect of Imperfections**

In tests on real bars, imperfections such as initial curvature or eccentricity of the load are inevitable. Some bending moment, however slight, must therefore be present as soon as any load is applied, and consequently bending proceeds from the beginning of loading. Consider a pinned elastic column with the load applied at a small distance $e$ from the centroidal axis. The deflection $v$ can easily be shown to be governed by

\[ EIv'' + Pv = Pe, \]

and the maximum deflection is found to be

\[ v_{max} = e \left[ \sec \left( \frac{\pi}{2} \sqrt{\frac{P}{P_E}} \right) - 1 \right]. \]

The deflection remains of the order of $e$ until the load gets close to the Euler load, when it begins to grow large.

A similar conclusion holds for a column with an initial deflection $v_0$ of amplitude $e$. The equation governing the total deflection $v$ under an axial load $P$ is

\[ EI(v'' - v_0'') + Pv = 0, \]

and if the column is pinned while $v_0$ is assumed as $v_0(x) = e \sin(\pi x/L)$, then the maximum deflection is

\[ v_{max} = \frac{e}{1 - P/P_E}. \]

For an imperfect column, then, the buckling load may be interpreted as the load in the vicinity of which the imperfections become significantly amplified. It is for this reason that the tangent-modulus load must necessarily appear as the buckling load of work-hardening elastic–plastic columns, since the possibility of remaining straight when this load is exceeded is open only to perfect columns.

In columns made of an elastic–perfectly plastic material, both the reduced and tangent moduli are zero, and therefore a perfect column, as discussed previously, will buckle elastically or yield in direct compression at supercritical and subcritical slenderness ratios, respectively. Such a sharp transition is, in fact, hardly ever observed in real columns made of structural steel, a material that is fairly well represented as elastic–perfectly plastic.
The deviation from theoretical behavior can also be ascribed to imperfections. An approximate theory, developed by several nineteenth-century authors and recently reviewed by Mortelhand [1987], results in the formula

\[
\frac{(1 - \sigma/\sigma_Y)\sigma/\sigma_Y}{1 - (1 + \eta)\sigma/\sigma_Y} = \frac{\pi^2 E}{\sigma_Y(L_e/r)^2},
\]

where \( \eta = Ahe/2I \), \( h \) being the beam depth and \( e \), as before, the amplitude of the initial deflection.

The effect of imperfections is of even greater significance in the analysis of post-bifurcation behavior. This topic has been extensively reviewed by Calladine [1973] and Hutchinson [1974].

**Other Uniaxial Buckling Problems: Rings and Arches**

From the preceding arguments we can infer that in all buckling problems in which uniaxial stress is assumed, the elastic solution can be used to give the buckling load with \( E \) replaced by \( E_t \), provided that \( E_t \) can be taken as constant. Take, for example, a circular ring under an external pressure \( q \) per unit length of center line, the radius of the center line being \( R \). The compressive force in the ring is thus \( qR \), and the stress is \( \sigma = qR/A \), where \( A \) is the cross-sectional area. The well-known solution is due to Bresse (see Timoshenko and Gere [1961]). The fundamental buckling mode is shown in Figure 5.3.2(a), with the ring deformed into an ellipse, and the critical pressure is

\[
q_{cr} = \frac{3EI}{R^3},
\]

where \( I \) is the second moment of area of the cross-section for bending in the plane of the ring.

![Figure 5.3.2](image)

**Figure 5.3.2.** Buckling of a ring or arch: (a) complete ring; (b) hinge-ended arch.

The four points on the buckled ring whose radial displacement is zero are the nodes of the buckling modes. Any half of the ring between two opposite
Chapter 5 / Problems in Plastic Flow and Collapse I

nodes is equivalent to a semicircular arch that is hinged at both ends, and therefore Bresse’s formula furnishes the critical buckling pressure for such an arch. More generally, for a hinged-ended arch subtending an angle $2\alpha$ the fundamental buckling mode can be expected to be as shown in Figure 5.3.2(b), and the result for the critical pressure is

$$q_{cr} = \left( \frac{\pi^2}{\alpha^2} - 1 \right) \frac{EI}{R^3}.$$  

These results may be immediately converted to the inelastic case. Limiting ourselves to the ring, we obtain

$$\frac{\sigma}{E_t(\sigma)} = \frac{3}{(R/r)^2},$$

where $r = \sqrt{I/A}$ as before. If the stress-strain relation is given by the generalized Prager formula with $n = 1$, an explicit formula for the stress, and hence for the critical pressure, is obtained:

$$q_{cr} = \frac{\sigma_\infty A/R}{1 + \left( \frac{\sigma_\infty}{3E} \right)(R/r)^2}.$$  

For a ring of rectangular cross-section, with depth $h$ (in the radial direction) and width $b$ (in the axial direction), we have $A = bh$ and $r^2 = h^2/12$. Defining $p = q/b$ as the pressure in the usual sense (per unit area), we obtain

$$p_{cr} = \frac{\sigma_\infty h/R}{1 + 4\left( \frac{\sigma_\infty}{E} \right)(R/h)^2},$$

a formula that coincides with that derived by Southwell [1915] when $\sigma_\infty$ is identified with the yield stress. Southwell’s result was actually intended for cylindrical tubes rather than rings, but based on an assumption of uniaxial stress — a highly questionable assumption for a shell, as shown in the next subsection.

5.3.3. Plastic Buckling of Plates and Shells

Introduction

Consider a flat plate subject to an applied in-plane force per unit length $F_a$ around its edge and a distributed in-plane force per unit area $p_a$. The membrane-force field $N_{\alpha\beta}$ obeys the equilibrium equation (5.2.3) and the boundary condition $\nu_\beta N_{\alpha\beta} = F_a$. The equilibrium becomes unstable if it is possible for the plate to undergo a deflection $w(x_1, x_2)$ with no additional forces applied. The membrane forces $N_{\alpha\beta}$ can be assumed to be related to the average Green–Saint-Venant strains $\bar{E}_{\alpha\beta}$ given by Equation (5.2.7), and
if the former do not change, then neither do the latter. The middle plane will therefore undergo a second-order displacement $\bar{u}_\alpha$ such that

$$2\bar{\varepsilon}_{\alpha\beta} = \bar{u}_{\alpha,\beta} + \bar{u}_{\beta,\alpha} = -w_{,\alpha} \, w_{,\beta}.$$  

The work done by the applied forces on this displacement is

$$\oint_C F^\alpha_\alpha \bar{u}_\alpha \, ds + \int_A p_\alpha \bar{u}_\alpha \, dA = \oint_C \nu_\beta N_{\alpha\beta} \bar{u}_\alpha \, ds + \int_A p_\alpha \bar{u}_\alpha \, dA$$

$$= \int_A [(N_{\alpha\beta,\beta} + p_\alpha) \bar{u}_\alpha + N_{\alpha\beta} \bar{u}_{\alpha,\beta}] \, dA$$

with the help of the two-dimensional divergence theorem. The quantity in parentheses vanishes as a result of (5.2.3). Since $N_{\alpha\beta}$ is symmetric, $N_{\alpha\beta} \bar{u}_{\alpha,\beta} = N_{\alpha\beta} \bar{\varepsilon}_{\alpha\beta} = -\frac{1}{2} N_{\alpha\beta} w_{,\alpha} \, w_{,\beta}$, and the second variation of the external potential energy can finally be obtained as

$$\delta^2 \Pi_{\text{ext}} = \int_A N_{\alpha\beta} \delta w_{,\alpha} \, \delta w_{,\beta} \, dA. \quad (5.3.15)$$

For $\delta^2 \Pi_{\text{int}}$ we have

$$\delta^2 \Pi_{\text{int}} = \int_A \delta M_{\alpha\beta} \delta w_{,\alpha\beta} \, dA. \quad (5.3.16)$$

Since

$$\delta M_{\alpha\beta} \delta w_{,\alpha\beta} = (\delta M_{\alpha\beta} \delta w_{,\alpha})_{,\beta} - \delta M_{\alpha\beta,\beta} \delta w_{,\alpha},$$

and since the edge conditions are usually such that $\nu_\beta \delta M_{\alpha\beta} \delta w_{,\alpha} = 0$ on $C$, the equation expressing the energy criterion becomes

$$\int_A (\delta M_{\alpha\beta,\beta} - N_{\alpha\beta} \delta w_{,\beta}) \delta w_{,\alpha} \, dA = 0.$$  

Support conditions on a plate are rarely such that the deflection itself is unconstrained; another integration by parts is usually necessary, leading to the differential equation

$$\delta M_{\alpha\beta,\alpha\beta} - (N_{\alpha\beta} \delta w_{,\beta})_{,\alpha} = 0. \quad (5.3.17)$$

Let the incremental relation between strain and stress for an isotropic material in a state of plane stress be written in the form

$$\dot{\varepsilon}_{\alpha\beta} = \frac{1}{E} [(1 + \nu)\bar{\sigma}_{\alpha\beta} - \nu \bar{\sigma}_{\gamma\gamma} \delta_{\alpha\beta}],$$

where $\nu$ is the instantaneous contraction ratio, and $E$ is the instantaneous modulus, in general not equal to the uniaxial tangent modulus discussed
before. If the tangent-modulus theory is applied to the plate problem, then an incremental moment-curvature relation can be written in the form

$$\delta M_{\alpha\beta} = \bar{D}[(1 - \bar{\nu}) \delta w_{\alpha\beta} + \bar{\nu} \delta w_{\gamma\gamma} \delta_{\alpha\beta}], \quad (5.3.18)$$

where $\bar{D} = \bar{E}h^3 / 12(1 - \bar{\nu}^2)$ is the effective plate modulus. The parameters $\bar{E}$ and $\bar{\nu}$, and hence also $\bar{D}$, are functions of $N_{\alpha\beta}$, and the general form of (5.3.17) when the relations (5.3.18) are substituted is complicated. In what follows, only examples with a uniform membrane-force field will be considered.

**Circular Plate Under Radial Load**

A simple example is that of a circular plate under a uniformly distributed compressive radial load applied around its edge. Let $a$ be the radius of the plate, $h$ its thickness, and $N$ the magnitude of the applied load per unit length of circumference. It is easy to see that a uniform state of plane stress, $\sigma_r = \sigma_\theta = -N/h$, is in equilibrium, satisfies the compatibility conditions if the plate is elastic, and obeys the yield criterion everywhere if it obeys it anywhere when the plate is plastic.

If the buckling is assumed axisymmetric, the deflection of the middle plane being $w(r)$, then the second integration by parts leading to (5.3.17) may be dispensed with, and the energy criterion can be expressed by the differential equation

$$\frac{1}{r}[(r \delta M_r)' - \delta M_\theta] + N\phi = 0,$$

where $\phi = \delta w'$ is the virtual rotation of the radial lines, with the prime denoting differentiation with respect to $r$. The axisymmetric form of (5.3.18) is

$$\delta M_r = \bar{D} \left( \frac{\phi'}{r} + \bar{\nu} \frac{\phi}{r} \right), \quad \delta M_\theta = \bar{D} \left( \frac{\phi}{r} + \bar{\nu} \phi' \right),$$

and the equation governing $\phi$ is thus

$$\phi'' + \frac{1}{r} \phi' + \left( \frac{N}{\bar{D}} - \frac{1}{r^2} \right) \phi = 0, \quad (5.3.19)$$

a Bessel equation of order 1. The general solution that is regular at $r = 0$ is $\phi(r) = J_1(\lambda r)$, where $\lambda = \sqrt{N/\bar{D}}$ and $J_1$ is the Bessel function of the first kind of order 1. Let $\lambda = k/a$ be the smallest nonzero root for which $\phi$ satisfies the boundary condition. The critical load is then $N_{cr} = \sigma_{cr} h$, where $\sigma_{cr}$ is, in view of the definition of $\bar{D}$, the solution of the nonlinear equation

$$\sigma = \frac{k^2 \bar{E}}{12 (1 - \bar{\nu}^2) \left( \frac{h}{a} \right)^2}, \quad (5.3.20)$$

$\bar{E}$ and $\bar{\nu}$ being, as noted above, functions of $\sigma$. 

If the edge of the plate is clamped, then the edge condition \( \phi(a) = 0 \) leads to the characteristic equation \( J_1(k) = 0 \), whose smallest nonzero root is \( k = 3.832 \). If the edge of the plate is free to rotate, then \( M_r \) must vanish there, so that the edge condition is \( \phi'(a) + \bar{\nu}\phi(a)/a = 0 \). The characteristic equation then becomes

\[
J_0(k) - \frac{1 - \bar{\nu}}{k} J_1(k) = 0.
\]

Except in the case of the elastic plate, for which \( \bar{\nu} = \nu \) (the Poisson’s ratio), this equation must be solved simultaneously with (5.3.20) in order to find \( \sigma_{cr} \).

It is now necessary to evaluate \( \bar{E} \) and \( \bar{\nu} \) as functions of \( \sigma \). We assume the plate material to be work-hardening and governed by the Mises criterion and its associated flow rule. The plastic strain rate can then be written as

\[
\dot{\varepsilon}^p_{ij} = \frac{9}{4} \frac{s_{kl}s_{kl}}{H\sigma_Y^2} s_{ij},
\]

where \( \sigma_Y \) is the current value of the uniaxial yield stress and \( H \) is the uniaxial plastic modulus (related to the tangent modulus by \( H_{\text{inv}} = E_{\text{t,inv}} - E_{\text{inv}} \)), as can easily be verified by the substitutions \( s_{11} = \frac{2}{3} \sigma, \ s_{22} = s_{33} = -\frac{1}{3} \sigma \), and \( |\sigma| = \sigma_Y \). In a state of plane stress,

\[
s_{kl}s_{kl} = \sigma_{\alpha\beta} \dot{\sigma}_{\alpha\beta} - \frac{1}{3} \sigma_{\alpha\alpha} \dot{\sigma}_{\beta\beta},
\]

and therefore, if currently \( \sigma_1 = \sigma_2 = \sigma \),

\[
\dot{\varepsilon}_1^p = \dot{\varepsilon}_2^p = \frac{1}{4H}(\dot{\sigma}_1 + \dot{\sigma}_2).
\]

The complete incremental stress-strain relations are therefore

\[
\dot{\varepsilon}_1 = \left( \frac{1}{E} + \frac{1}{4H} \right) \dot{\sigma}_1 - \left( \frac{\nu}{E} - \frac{1}{4H} \right) \dot{\sigma}_2,
\]

\[
\dot{\varepsilon}_2 = \left( \frac{1}{E} + \frac{1}{4H} \right) \dot{\sigma}_2 - \left( \frac{\nu}{E} - \frac{1}{4H} \right) \dot{\sigma}_1.
\]

Thus

\[
\frac{1}{E} = \frac{1}{E} + \frac{1}{4H} = \frac{1}{4E_t} + \frac{3}{4E}
\]

and

\[
\bar{\nu} = \frac{1 + 4\nu - E/E_t}{3 + E/E_t}.
\]

It should be noticed that for small values of \( E_t/E \), \( \bar{\nu} \approx -1 + 4(1 + \nu)E_t/E \) and \( \bar{E} \approx 4E_t \), so that the factor \( \bar{E}/(1 - \bar{\nu}^2) \) in Equation (5.3.20) tends to a constant fraction, \( \frac{1}{2}(1-\nu) \), of the elastic value. This result, which is similar to
what occurs in other plate and shell buckling problems, is quite unreasonable
when compared with the uniaxial case, and indeed with experimental data.
Furthermore, the result is not limited to the Mises criterion but would also be
produced by any isotropic yield criterion that is smooth at \( \sigma_1 = \sigma_2 = \pm \sigma_Y \),
since all such yield loci must be tangent there. We are left with the conclusion
that incremental plasticity with a smooth yield surface may not be applicable
to the analysis of multiaxial instability problems.

Considerable improvement is obtained when the deformation theory of
plasticity discussed in 3.2.1 is used. The Hencky theory, in particular, is
based on the Mises criterion, and gives the plastic strain as

\[
\varepsilon_{ij}^p = \frac{3}{2\bar{\sigma}} \sigma_{ij},
\]

(5.3.21)

where

\[
\bar{\sigma} = \sqrt{\frac{3}{2}} \sigma_{ij} \sigma_{ij}, \quad \varepsilon_p = \sqrt{\frac{2}{3}} \varepsilon_{ij}^p
\]

are the equivalent stress and plastic strain, related to each other by the
uniaxial relation. The incremental form of (5.3.21) is

\[
d\varepsilon_{ij}^p = \frac{3}{2\bar{\sigma}} \left[ \left( \frac{d \varepsilon_p}{\bar{\sigma}} - \frac{\bar{\sigma}}{\bar{\sigma}} d \bar{\sigma} \right) s_{ij} + \varepsilon_p^p d s_{ij} \right].
\]

Upon introducing the uniaxial secant and tangent moduli \( E_s \) and \( E_t \), defined by

\[
\frac{1}{E_s} = \frac{1}{E} + \frac{\varepsilon_p^p}{\bar{\sigma}}, \quad \frac{1}{E_t} = \frac{1}{E} + \frac{d \varepsilon_p}{d \bar{\sigma}},
\]

the incremental relation may be written as

\[
d\varepsilon_{ij}^p = \frac{3}{2} \left[ \left( \frac{1}{E_t} - \frac{1}{E_s} \right) \frac{d \bar{\sigma}}{\bar{\sigma}} s_{ij} + \left( \frac{1}{E_s} - \frac{1}{E} \right) d s_{ij} \right].
\]

Applying the relation to the plane-stress case with \( \sigma_1 = \sigma_2 \) leads to the
instantaneous modulus and contraction ratio

\[
\frac{1}{\bar{E}} = \frac{1}{4E_t} + \frac{3}{4E_s}
\]

and

\[
\bar{\nu} = -\frac{E_t/4E_t + 2(1 - 2\nu) - 3E/E_s}{E/E_t + 3E/E_s}.
\]

For a gradually flattening uniaxial stress-strain curve, when \( E_t \ll E_s \ll E \)
we have \( \bar{E} \approx 4E_t \) as in the incremental theory, but \( \bar{\nu} \approx -(1 - 6E_t/E_s) \), and
therefore \( \bar{E}/(1 - \bar{\nu}^2) \approx E_s/3 \). To the first approximation, then, it is the
secant modulus, rather than the elastic modulus, that governs the buckling,
resulting in a much smaller critical load than that given by the incremental
theory.
An explanation of the failure of incremental theory based on an isotropic yield criterion to predict a reasonable buckling load is due to Phillips [1972], who points out that the yield criteria of work-hardening materials become anisotropic almost immediately upon plastic loading.

**Torsional Buckling of a Cruciform Column**

If a column is made up of thin plate sections that do not form a closed tube and is sufficiently short, then under the action of a compressive axial load it will buckle by twisting rather than bending. Consider, for example, the cross-shaped column shown in Figure 5.3.3, and in particular the flange whose middle plane is the xy-plane with y positive. If the virtual twist angle per unit length of the cross-section at \( x \) is \( \phi(x) \), then the virtual deflection at \( (x, y) \) is \( \delta w(x, y) = y\phi(x) \). The axial load \( P \) may be assumed to be uniformly distributed with intensity \( P/4b \) per unit width of flange. The second-order external potential energy on a fiber of width \( dy \) is therefore

\[
d(\delta^2\Pi_{ext}) = -\frac{P}{4b}dyy^2\int_0^L \phi'^2 dx,
\]

and, for the whole flange,

\[
\delta^2\Pi_{ext} = \int_{y=0}^{y=b} d(\delta^2\Pi_{ext}) = -\frac{Pb^2}{12} \int_0^L \phi'^2 dx.
\]

The energy criterion may be expressed by adding this quantity to \( \delta^2\Pi_{int} \) as given by (5.3.16) and equating the result to zero. Now

\[
\delta w_{11} = y\phi'', \quad \delta w_{12} = \phi', \quad \delta w_{22} = 0,
\]

so that

\[
\delta M_{\alpha\beta} \delta w_{\alpha\beta} = \bar{D}[y^2\phi'^2 + (1-\bar{\nu})\phi'^2].
\]

However, \( \phi'' \) is of order \( \phi'/L \), and, if \( L \gg b \), then the first term in brackets may be neglected in comparison with the second. Consequently, independently of the form taken by \( \phi' \),

\[
P_{cr} = \frac{12\bar{D}(1-\bar{\nu})}{b},
\]

or, because of the definition of \( \bar{D} \), we can define \( \sigma_{cr} = P_{cr}/4bh \) as the solution of

\[
\sigma = \bar{G} \left( \frac{h}{b} \right)^2 \bar{G} \frac{(1-\bar{\nu})}{(1+\bar{\nu})},
\]

where \( \bar{G} = \bar{E}/(1+\bar{\nu}) \) is the instantaneous shear modulus, \( d\tau/d\gamma \). Note that the result is independent of the column length \( L \).

In any incremental theory with a yield criterion of the form (3.3.5) and an associated flow rule, when \( \tau = 0 \) the normal to the yield locus is directed
Figure 5.3.3. Cruciform (cross-shaped) column: initial and buckled geometry.

in the $\sigma$-direction, and therefore $d\gamma^p = 0$. Consequently $\bar{G} = G$, so that the buckling load is unaffected by plasticity — again an untenable result.

On the other hand, in the Hencky theory we have

$$d\gamma^p = 3 \left( \frac{1}{E_s} - \frac{1}{E} \right) d\tau,$$

so that

$$\bar{G} = \frac{E_s}{3 + (1 - 2\nu)E_s/E}.$$

Once again, it is primarily the secant modulus that determines the critical load. Experiments by Gerard and Becker [1957] on aluminum columns show very good agreement with the prediction of deformation theory for $\nu = \frac{1}{2}$ (see Figure 5.3.4).

Shell Under External Pressure

The behavior of shells is in general much more complicated than that of plates. However, the buckling behavior of certain thin-walled shells may be studied by means of a simplified theory, known as the Donnell–Mushtari–Vlasov (DMV) theory, whose structure is essentially the same as that of plate theory (see Niordson [1985], Chapter 15). It must be kept in mind,
though, that shells are far more imperfection-sensitive than bars or plates; even the slightest imperfections can greatly reduce the buckling load.

For an elastic thin-walled cylindrical tube of mean radius $R$ and thickness $h$, subject to an external pressure $p$, the critical pressure can be found from the Bresse formula for the ring by substituting $D$ for $EI$, resulting in

$$p_{cr} = \frac{E}{4(1-\nu^2)} \left( \frac{h}{R} \right)^3.$$

The substitution is equivalent to assuming that the axial strain is zero. In other words the tube, when viewed axially, is in a state of plane strain as opposed to the plane-stress state of the ring. The axial stress is thus $\sigma_z = \nu \sigma_\theta$, where $\sigma_\theta = -pR/h$ is the circumferential stress, and the circumferential stress-strain relation is $\sigma_\theta = E \varepsilon_\theta/(1 - \nu^2)$.

In an elastic–plastic tube governed by incremental theory, the plane-strain condition can only be enforced incrementally, that is, $d\sigma_z = \nu \, d\sigma_\theta$. If buckling occurs after yielding, then the state of stress just before buckling is not known a priori, but must be determined by integrating the incremental relation, and checking at each step whether the buckling criterion is met.

The situation is somewhat simpler with deformation theory. The condition $\varepsilon_z = 0$ results in

$$\sigma_z = \frac{1}{2} \left[ 1 - (1 - 2\nu) \frac{E_s}{E} \right] \sigma_\theta.$$
The incremental relation is
\[ d\sigma_z = \frac{1}{2} \left[ 1 - (1 - 2\nu) \frac{E_s}{E} \right] d\sigma_\theta - \frac{1 - 2\nu}{2E} \frac{dE_s}{d\sigma} d\bar{\sigma}. \]

But
\[ \frac{dE_s}{d\sigma} = -\frac{E_s}{\sigma} \left( \frac{E_s}{E_t} - 1 \right) \]

and
\[ d\bar{\sigma} = d\sqrt{\sigma_\theta^2 - \sigma_\theta \sigma_z + \sigma_z^2} = \frac{(2\sigma_\theta - \sigma_z) d\sigma_\theta + (2\sigma_z - \sigma_\theta) d\sigma_z}{2\sigma}. \]

Thus both \( d\bar{\sigma} \) and \( d\sigma_z \) can be expressed as multiples of \( d\sigma_\theta \), and the expressions can be substituted in
\[ d\varepsilon_\theta = \frac{1}{E} (d\sigma_\theta - \nu d\sigma_z) + \frac{1}{2} \left( \frac{1}{E_s} - \frac{1}{E} \right) \frac{2\sigma_\theta - \sigma_z}{\bar{\sigma}} d\bar{\sigma}, \]
resulting in
\[ d\varepsilon_\theta = \frac{1}{\tilde{E}} d\sigma_\theta, \]

where \( \tilde{E} \) is in general a complicated function of \( \sigma_\theta \); it is this \( \tilde{E} \) that replaces \( E \) in the Bresse formula. When \( \nu = \frac{1}{2} \), however, \( \tilde{E} \) turns out to be just \( 4E_t/3 \), where \( E_t \) is the uniaxial tangent modulus. In this case the same result is given by incremental plasticity theory.

The problem of the spherical shell is easier, since the state of stress before buckling is \( \sigma_\theta = \sigma_\phi = -\sigma \), where \( \sigma = pR/2h \). The critical value of \( \sigma \) for elastic buckling is given by
\[ \sigma_{cr} = \frac{E}{\sqrt{3(1-\nu^2)}} \frac{h}{R}. \]

The value for plastic buckling can be obtained by solving
\[ \sigma = \frac{\tilde{E}}{\sqrt{3(1-\tilde{\nu}^2)}} \frac{h}{R}, \]

where \( \tilde{E} \) and \( \tilde{\nu} \) are the same functions of \( \sigma \) as are used for the circular plate under radial force. Figure 5.3.5 shows the critical stress for the buckling of a spherical shell, based on both the incremental and deformation theories, using the Ramberg–Osgood formula with \( \alpha = 0.1 \) and \( m = 6 \). For comparison, the stress-strain curve is also shown.

**Exercises: Section 5.3**

1. Find the reduced modulus \( E_r \) in terms of the elastic modulus \( E \) and the tangent modulus \( E_t \) for (a) an ideal sandwich section and (b) a thin-walled square tube section. Compare with the result for a rectangular section when \( E_t/E = 0.1 \) and when \( E_t/E = 0.02 \).
2. Plot column curves based on both the tangent-modulus and the reduced-modulus theories (for a rectangular section) for (a) the generalized Prager formula with \( n = 1 \), (b) the generalized Prager formula with \( n = 4 \), (c) the Ramberg–Osgood formula with \( \alpha = 0 \) and \( m = 4 \).

3. For a ring under radial pressure, plot suitably nondimensionalized buckling curves (\( q_{cr} \) against \( R/r \)) on the basis of (a) the Prager formula and (b) the Ramberg–Osgood formula with \( \alpha = 0.2 \) and \( m = 6 \).

4. Is it possible to find an instantaneous plate modulus \( \bar{D} \) and a contraction ratio \( \bar{\nu} \) for the buckling problem of a circular plate under a compressive radial load when the plate material obeys the incremental theory of plasticity with the Tresca criterion and its associated flow rule? Explain.

5. Perform the analysis leading to the results shown in Figure 5.3.5.