Chapter 6

Problems in Plastic Flow and Collapse II
Applications of Limit Analysis

Introduction

The theorems of limit analysis for standard elastic–perfectly plastic three-dimensional continua in arbitrary states of deformation were proved in 3.5.1. The proof, due to Drucker, Prager, and Greenberg [1952],\(^1\) was the final link in a chain of development of the theory, which began with proofs for beams and frames by Gvozdev [1938], Horne [1950], and Greenberg and Prager [1951], followed by a proof for bodies in plane strain by Drucker, Greenberg, and Prager [1951].

In this chapter applications of the theorems are presented. Section 6.1 deals with plane problems in both plane strain and plane stress. Section 6.2 deals with beams under combined loading (including arches), Section 6.3 with trusses and frames, and Section 6.4 with plates and shells.

It is shown that as a rule, plausible velocity fields are easier to guess than stress fields, and therefore in many cases only upper-bound estimates are available. Of particular importance are velocity fields called mechanisms, in which deformation is concentrated at points, lines, or planes, with the remaining parts of the system moving as rigid bodies. The use of mechanisms for estimating collapse loads antedates the development of plasticity theory. Examples include Coulomb’s method of slip planes for studying the collapse strength of soil, the plastic-hinge mechanism due to Kazinczy [1914] for steel frames, and the yield-line theory of Johansen [1932] for reinforced-concrete slabs, later extended to plates in general.

\(^1\)But already given in the book by Prager and Hodge [1951].
Section 6.1 Limit Analysis of Plane Problems

As pointed out in Section 5.1, slip-line theory as a rule gives only upper bounds, unless an admissible extension of the stress field from the region covered by the slip-line field to the rest of the body is found. A convenient method for finding lower bounds is by means of discontinuous stress fields, some examples of which were studied in Section 5.1.

6.1.1. Blocks and Slabs with Grooves or Cutouts

Among the earliest applications of limit analysis to plane problems are studies of the effects of cutouts on the yielding of rectangular slabs or blocks subject to tension perpendicular to a side. In the absence of cutouts, the slab may be assumed to collapse under a uniform state of tensile stress $\sigma = \sigma_Y = 2k$, the Tresca criterion being assumed in plane stress. The cutouts may be expected to reduce the average or nominal tensile stress $\sigma$ required for collapse to $2\rho k$ ($0 < \rho < 1$), where $\rho$ is called the cutout factor. Limit analysis may be used to find bounds on the cutout factor.

Tension of a Grooved Rectangular Block

The following example, illustrated in Figure 6.1.1, is discussed by Prager and Hodge [1951]. A statically admissible stress field is shown in Figure 6.1.1(a). The trapezoidal regions on either side of the line of symmetry $OO$ correspond to the truncated wedge of Figure 5.1.6(e) (page 289), except that the stresses are tensile rather than compressive. The rectangular regions beyond the trapezoids are in simple tension with stress $\sigma = 2k(1 - \sin \gamma)$, where $\gamma$ is the wedge semi-angle. In order to produce the best lower bound, $\gamma$ should be as small as possible, subject to the geometric restriction discussed in connection with Figure 5.1.6(e) and the evident additional restriction that the flanks of the wedge be tangent to the cutout circles. It can be shown, the details being left to an exercise, that these restrictions require that $2\sin 2\gamma = 1 + \sin \gamma$. The smallest angle obeying this equation is $0.379$ radian, so that $\sin \gamma = 0.370$, and the lower bound to the cutout factor is $0.630$.

A simple kinematically admissible velocity field is shown in Figure 6.1.1(b). The portion above the line $AB$ slides rigidly along this line with respect to the portion below. If the sliding speed is $v$, then the plastic dissipation per unit area along $AB$ is $kv$, while the length of the line is $3a\sqrt{2}$. The external rate of work per unit thickness is $\sigma \cdot 4a \cdot v/\sqrt{2}$. Equating this rate to the total internal dissipation per unit thickness, $3\sqrt{2}kva$, yields the upper bound $\sigma = 3k/2$, or $\rho = 3/4$.

Figure 6.1.1(c) shows a slip-line field in which the region bounded by the slip lines $AB$ and $A'B$ (as well as its mirror image) is in a state of
axisymmetric stress as studied in Section 4.4. The stress $\sigma_\theta$ in this region is

$$\sigma_\theta = 2k \left( 1 + \ln \frac{r}{a} \right),$$

where $r$ is measured from the center $O$ of the circle. The resultant axial force per unit thickness is therefore

$$F = 2 \int_a^{2a} \sigma_\theta \, dr = 4k \int_a^{2a} \left( 1 + \ln \frac{r}{a} \right) \, dr = 8ka \ln 2.$$

The nominal stress is thus $\sigma = F/4a = 2k \ln 2$, corresponding to a cutout factor of $\ln 2 = 0.693$. Since the stress field has not been extended to the regions outside the slip-line field, this value cannot be a lower bound. On the other hand, the slip-line field implies a solution for the velocity field, which, together with the rigid axial motion of the regions outside it, forms a kinematically admissible velocity field for the body as a whole, and therefore the result gives an improved upper bound for the cutout factor. The bounds on the cutout factor are therefore

$$0.630 \leq \rho \leq 0.693.$$

Tension of a Square Slab With a Slit

A thin square slab with a narrow slit perpendicular to the direction of the load is shown in Figure 6.1.2(a). The slit width is assumed to be of
the same order of magnitude as the slab thickness, both being much smaller than the sides of the square. A kinematically admissible velocity field may be based on a shearing failure mode as shown in Figure 6.1.2(b). If the shearing plane makes an angle $\alpha$ with the load direction then the total area of the surface of sliding is $(1 - \beta)ah\csc \alpha$, where $h$ is the slab thickness; and if the relative velocity of motion of the two parts of the slab is $v$, then the plastic dissipation per unit area is $kv \sec \alpha$. The total internal dissipation is therefore $2k(1 - \beta)ahv \csc 2\alpha$. The external work rate is $\sigma ahv$, and therefore an upper bound to the cutout factor is $(1 - \beta) \csc 2\alpha$. The best upper bound is obtained for $\alpha = \pi/4$, and gives $\rho = 1 - \beta$.

A discontinuous stress field for the determination of a lower bound was constructed by Hodge [1953] (see Hodge [1959], Section 12-2), and is shown in Figure 6.1.2(c). Since the problem is one of plane stress and not plane strain, it is not the Prager jump conditions (5.1.5) but the more general jump conditions (5.1.6), together with the yield criterion, that must be used. There being four distinct regions, the total number of unknown stress variables ($n_i, r_i, \theta_i$) ($i = 1, 2, 3, 4$) is 12. The equations for these variables are furnished by, first, two traction boundary conditions each on two sides of the square and on the face of the slit, and second, by two jump conditions each on the boundaries between regions 1 and 2, 2 and 3, and 2 and 4. In particular, the fact that $\tau_{xy}$ vanishes on the external boundaries of regions 1, 3 and 4 means that it vanishes throughout these regions. Regions 3 and 4 can reasonably be assumed to be in a state of simple tension and compression, respectively, so that $\theta_3 = \theta_4 = 3\pi/4$. In region 1, $\theta$ is either $3\pi/4$ or
\( \pi/4; \) it will be assumed that \( \sigma_y \geq \sigma_x \) there, so that \( \theta_1 = \pi/4 \). The remaining boundary conditions accordingly give

\[
n_1 - r_1 = \sigma, \quad n_3 - r_3 = 0, \quad n_4 + r_4 = 0,
\]

so that \( n_1, n_3 \) and \( n_4 \) can be eliminated. The jump conditions are

\[
\begin{align*}
n_2 + r_2 \sin 2(\theta_2 - \chi_{12}) &= \sigma + r_1 \cos 2\chi_{12}, \\
n_2 + r_2 \sin 2(\theta_2 - \chi_{23}) &= r_3(1 - \cos 2\chi_{23}), \\
n_2 + r_2 \sin 2(\theta_2 - \chi_{24}) &= -r_4(1 + \cos 2\chi_{24}), \\
r_2 \cos 2(\theta_2 - \chi_{12}) &= r_1 \sin 2\chi_{12}, \\
r_2 \cos 2(\theta_2 - \chi_{23}) &= -r_3 \sin 2\chi_{23}, \\
r_2 \cos 2(\theta_2 - \chi_{24}) &= -r_4 \sin 2\chi_{24},
\end{align*}
\]

where the angles \( \chi_{12}, \chi_{23} \) and \( \chi_{24} \), giving the inclinations of the normals to the boundary lines with the \( x \)-axis, can be expressed in terms of \( \beta \) and \( \xi \). Once the equations are solved, satisfaction of the yield inequality

\[
2r + |n - r| + |n - r| \leq 4k
\]

in each region produces four inequalities on the cutout factor \( \rho = \sigma/2k \):

\[
\begin{align*}
\rho\beta &\leq 1 - \xi, \quad \rho \leq 1 - \beta, \quad \rho\beta \leq \xi, \\
\rho\sqrt{[\xi + \beta(1 - \beta)]^2 + 4\beta^2} &\leq \beta + \xi(1 - \beta).
\end{align*}
\]

If the second inequality were satisfied as an equality, then the result \( \beta = 1 - \rho \) would coincide with the upper bound. It turns out that if the equality is assumed, then the remaining inequalities are obeyed if \( \xi = 1 - \beta + \beta^2 \). In this problem, then, the exact cutout factor has been found. Other problems involving slabs with cutouts were treated by Hodge and various coworkers (see Hodge [1959], Chapter 12, for a survey).

6.1.2. Problems in Bending

Pure Bending of a Notched Bar

The notched bar shown in Figure 6.1.3(a) is of rectangular cross-section and subject to equal and opposite couples \( M \) applied at its ends. The discontinuous stress field shown is statically admissible, and plastically admissible in classical plane strain as well as for the Tresca criterion in plane stress. It is, of course, equivalent to the limiting stress distribution in a perfect beam, limited to the material below the notch, and gives a lower bound of \( \frac{1}{2}kba^2 \) for the ultimate moment, where \( b \) is the width of the beam.
An upper bound can be found on the basis of a mechanism, shown in Figure 6.1.3(b), in which the outer portions of the bar rotate rigidly by sliding along the arcs ACB and ADB, the inner region ACBD remaining stationary; the arcs must accordingly be circular. The mechanism resembles the plastic hinge discussed in 4.4.2.

If the angular velocity of rotation is $\omega$ and the radius of the arcs is $r$, then the plastic dissipation per unit area of sliding surface is $k r \omega$. If the angle subtended by the arcs is $2\alpha$, then the total area of the sliding surfaces is $4br\alpha$. But $r = \frac{L}{2} \csc \alpha$, so that the total internal dissipation is $k \omega ba^2 \alpha \csc^2 \alpha$. Equating this to the external work rate $2M \omega$ gives the upper bound $M = \frac{1}{2} kba^2 \alpha \csc^2 \alpha$. The smallest value of this occurs when $\tan \alpha = 2 \alpha$, and gives the upper bound of $M = 0.69kba^2$. For a bar that is wide enough to be regarded as bending in plane strain, A. P. Green [1953] found a slip-line field that gives the improved upper bound of $M = 0.63kba^2$.

**End-Loaded Cantilever in Plane Strain and Plane Stress**

Another problem studied by Green [1954a] by means of slip-line theory concerns the impending collapse of an end-loaded wide tapered cantilever, shown in Figure 6.1.4.¹ If the taper is not extreme and if the ratio $L/h$ of the length to the least depth is sufficiently great, it is reasonable to suppose that the collapse mechanism is of the plastic-hinge type. The slip-line fields shown in Figures 6.1.4(a) and (b) produce deformations similar to those

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¹For the prismatic beam, the problem was also studied by Onat and Shield [1955].
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corresponding to a hinge mechanism. In (a), the rigid portion to the right of
the slip-line field rotates about point $Y$; in (b) it slides over the circular arc
$PYQ$, as in the preceding problem, and $APYQB$ is a continuous slip line
that is a locus of velocity discontinuity.

![Diagram](a)

![Diagram](b)

![Diagram](c)

![Diagram](d)

**Figure 6.1.4.** End-loaded tapered cantilever: (a)–(b) slip-line fields; (c) sta-
tically admissible stress field. (d) Weakly supported prismatic cantilever.

A complete solution is found, with the help of the discontinuous but
statically and plastically admissible stress field shown in Figure 6.1.4(c), for
the beam with $15^\circ$ taper ($\theta = 75^\circ$), in which case $F_U = kbh$ for all values
of $L/h$. For greater values of $\theta$, no extension of the stress field into
the rigid regions is given and therefore the calculated value of $F$ is an upper
bound to $F_U$, though, as Green argues, it is likely to be very close, since
the proposed slip-line field shows remarkable similarity with the plastically
deformed region observed in experiments. For $\theta \geq 75^\circ$ the load is given, in
accordance with the slip-line field in (a), by

$$F = 2kbh \sin 2\theta \quad \text{for} \quad \frac{L}{h} \geq \frac{1}{2} \tan \theta (\sin 2\theta - \cos 2\theta).$$
For a given \( \theta \), as \( L/h \) is decreased below the limiting value, \( F \) increases.

For prismatic beams, Green [1954a,b] also constructed solutions for uniformly distributed loading, and for other boundary conditions, including “weakly” supported cantilevers (see Figure 6.1.4(d)) as well as beams fixed at both ends.

A much simpler velocity field for a prismatic beam is shown in Figure 6.1.5(a). In this picture region 1 undergoes simple shearing, regions 2 and 3 rotate rigidly, and regions 4 and 5 undergo tension or compression. In the last-named regions the vertical velocity does not vanish, so that the condition of no motion at the wall is not satisfied. A modification of the mechanism that satisfies this condition, shown in Figure 6.1.5(b), was proposed by Drucker [1956a]; here region 6 does not displace. The mechanism of Figure 6.1.5(a) is equivalent (with the direction of the force reversed) to that of Figure 6.1.5(c) for a center-loaded simply supported beam. Calculations are done here only for mechanism (a).

\[ \dot{\epsilon}_x + \dot{\epsilon}_y = 0 \] in every region, so that the mechanism applies to plane strain.
as well as to plain stress. For either the Mises or the Tresca criterion, the dissipation $D_p(\dot{\varepsilon})$ equals $\tau Y \dot{\Delta}/L$ in region 1 and $2\tau Y \dot{\Delta}/L$ in regions 4 and 5. Equating the external work rate to the total internal dissipation,

$$F \dot{\Delta} = \tau Y \frac{\dot{\Delta}}{L} \cdot b L d + 2\tau Y \frac{\dot{\Delta}}{L} \cdot b \left(\frac{h - d}{2}\right)^2,$$

leads to the upper bound

$$F = \tau Y b h \left[\frac{d}{h} + \frac{h}{2L} \left(1 - \frac{d}{h}\right)^2\right]. \quad (6.1.1)$$

The upper bound can be optimized by minimizing with respect to $d$. The minimum occurs at $d = h - L$ and leads to $F = \tau Y b h (1 - L/2h)$; but this result is valid only for $L < h$. For $L \geq h$, $d = 0$, that is, the mechanism is a plastic hinge consisting of two deforming triangles, and $F = \tau Y b h^2/2L$ — a result that, in plane stress, agrees with that of elementary beam theory for the Tresca criterion.

**End-Loaded Cantilever in Plane Stress: Lower Bound**

Drucker [1956a] also constructed a lower bound for the problem by means of the following statically admissible stress field:

$$\sigma_x = \frac{F x}{bh h} \frac{2\alpha^2}{1 - \cos \alpha} \cos \left[\alpha \left(1 - \frac{2|y|}{h}\right)\right] \text{sgn } y,$$

$$\tau_{xy} = \frac{F \alpha}{bh h} \frac{1 - \cos \alpha}{1 - \cos \alpha} \sin \left[\alpha \left(1 - \frac{2|y|}{h}\right)\right], \quad \sigma_y = 0,$$

with $\alpha \leq \pi/2$. The yield criterion (3.3.5) is met at $x = L$ if

$$\alpha = \frac{h \sigma_Y}{2L \tau_Y} \quad \text{and} \quad F = \tau Y b h \frac{1 - \cos \alpha}{\alpha}. \quad (6.1.2)$$

In accordance with the limitation on $\alpha$, the result is limited to $h/L \leq \pi \tau_Y/\sigma_Y$. It can easily be ascertained that for small values of $h/L$, the result for $F$ approaches $\sigma_Y b h^2/4L$ as in the beam-theory approach.

For $h/L > \pi \tau_Y/\sigma_Y$, the stress field is

$$\sigma_x = 0, \quad \sigma_y = 0, \quad \tau_{xy} = \tau_Y \quad \text{for } |y| < \frac{d}{2},$$

and

$$\sigma_x = \sigma_Y \frac{x}{L} \cos \left[\frac{\pi}{2} \left(1 - \frac{2|y| - d}{h - d}\right)\right] \text{sgn } y, \quad \sigma_y = 0,$$

\(^1\)Drucker [1956a] considered a Tresca material only (i.e., $\sigma_Y = 2\tau_Y$). The extension to the more general yield criterion (3.3.5) is straightforward.
\[ \tau_{xy} = \tau_Y \sin \left[ \frac{\pi}{2} \left( 1 - \frac{2|y| - d}{h - d} \right) \right] \quad \text{for } |y| > \frac{d}{2}. \]

The stress field is in equilibrium if \( d = h - \pi \tau_Y L / \sigma_Y \). The load is then

\[ F = \frac{2}{\pi} \tau_Y bh \left[ 1 + \frac{d}{h} \left( \frac{\pi}{2} - 1 \right) \right] = \tau_Y bh \left[ 1 - (\pi - 2) \frac{\tau_Y L}{\sigma_Y h} \right]. \quad (6.1.3) \]

The upper and lower bounds are compared in Figure 6.1.6.

\[ \frac{2FL}{\tau_Y bh^2} \]

\( \sigma_Y = 2 \)

\( \tau_Y \) (Tresca)

Upper bound

Lower bound

Figure 6.1.6. Upper and lower bounds for an end-loaded prismatic cantilever beam

**I-Beams**

Green [1954b] extended his results for prismatic beams to I-beams by assuming that the slip-line fields derived for rectangular beams in plane stress prevail in the web, while the flanges are in pure tension or compression. The shear force is thus carried entirely by the web, while the bending moment is the sum of that provided by the web and the couple formed by the flange forces.

The simple velocity fields shown in Figure 6.1.5 were proposed for short I-beams by Leth [1954] with \( d \) as the actual web depth. The upper-bound load based on the field in (a) or (c) is

\[ F = \tau_Y \left( A_w + 2A_f \frac{h - d}{L} \right), \]

where \( A_w = t_w d \) is the web area and \( A_f = b_f (h - d) \) is the flange area, \( t_w \) and \( b_f \) being respectively the web thickness and flange width. Since \( A_f \) and \( A_w \) are typically of the same order of magnitude, the second term in parentheses may usually be neglected, and the upper-bound loaded may be approximated as \( \tau_Y A_w \).
For sufficiently long I-beams, however, it is reasonable to assume that failure is by a plastic-hinge mechanism, with \( F = M_U/L \) as given by elementary beam theory. For an I-beam,

\[
M_U = \frac{\sigma_Y}{4} [A_w d + 2A_f (h + d)] = \frac{\sigma_Y h}{4} (A_w + 4A_f),
\]

the approximate expression being based on the assumption that \( h - d \ll h \). Comparing the two upper bounds, then, we find that the plastic-hinge mechanism furnishes the lower one for

\[
\frac{L}{h} \geq \frac{\sigma_Y}{4\tau_Y} \left(1 + 4\frac{A_f}{A_w}\right),
\]

approximately, or about \( 5/2 \) for a beam made of Tresca material with \( A_f = A_w \). Additional results in the limit analysis of beams, derived on the basis of local behavior, are discussed in Section 6.2 (see 6.2.4).

### 6.1.3. Problems in Soil Mechanics

#### Yield Criterion and Flow Rule

The most commonly used yield criterion for soils is the Mohr–Coulomb criterion discussed in 3.3.3,

\[
\sigma_{\text{max}} - \sigma_{\text{min}} + (\sigma_{\text{max}} + \sigma_{\text{min}}) \sin \phi = 2c \cos \phi,
\]

where \( c \) is the cohesion and \( \phi \) is the angle of internal friction. The Mohr–Coulomb criterion includes, as limiting cases, (1) the Tresca criterion (with \( \phi = 0 \) and \( c = k \)), used to describe, for example, saturated clays, and (2) the cohesionless friction model (\( c = 0 \)) for cohesionless soils (dry sands and gravels).

If the material is taken as standard, then the flow rule at a regular point of the yield surface is

\[
\varepsilon_{\max}^p = \dot{\lambda}(1 + \sin \phi), \quad \varepsilon_{\min}^p = -\dot{\lambda}(1 - \sin \phi), \quad \varepsilon_{\text{int}}^p = 0.
\]

The flow rule implies a constant dilatancy ratio, defined as \( (\dot{\varepsilon}_1^p + \dot{\varepsilon}_2^p + \dot{\varepsilon}_3^p)/\dot{\varepsilon}_{\max}^p \), which at a regular point is given by \( (\dot{\varepsilon}_{\max}^p + \dot{\varepsilon}_{\min}^p)/(\dot{\varepsilon}_{\max}^p - \dot{\varepsilon}_{\min}^p) \) and is therefore equal to \( \sin \phi \). The measured dilatancy of most soils (as well as rocks and concrete) is usually significantly less than this,\(^1\) except in the case of undrained clays in which both internal friction and dilatancy are negligible. Most such materials, therefore, cannot be modeled as standard.

In a nonstandard model, the flow rule may be taken in the same form as above, but with the dilatancy angle \( \psi \) replacing \( \phi \).

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\(^1\)The dilatancy ratio rarely exceeds 0.1, while the angle of internal friction may be as high as 45° in dense, well-graded soils with angular particles.
The plastic dissipation in a standard Mohr–Coulomb material was shown by Drucker [1953] to be

\[ D_p(\dot{\varepsilon}^p) = c \cot \phi (\dot{\varepsilon}_1^p + \dot{\varepsilon}_2^p + \dot{\varepsilon}_3^p) \]

at any point of the yield surface, including the corners.

In plane plastic flow with \( \dot{\varepsilon}_3 = 0 \), as was pointed out in 3.3.4, \( \sigma_3 \) is the intermediate principal stress, even if \( \psi \neq \phi \). The criterion therefore takes the form

\[ |\sigma_1 - \sigma_2| + (\sigma_1 + \sigma_2) \sin \phi = 2c \cos \phi. \]

**Mechanisms**

A *Coulomb mechanism* in plane strain is one in which polygonal blocks of material move rigidly relative to one another. The interfaces between the blocks may be regarded as very narrow zones in which the strain rates are very large. In a nondilatant material, only shearing takes place in such a zone, so that the movement is one of sliding, and the interfaces are just slip lines, as discussed in Section 5.1. In the presence of dilatancy, dilatation as well as shearing takes place, and the movement involves separation in addition to sliding.\(^1\) In fact, it can easily be shown (the details are left to an exercise) that the velocity discontinuity forms an angle equal to \( \psi \) with the interface – that is, if the magnitude of the discontinuity is \( v \) then the sliding speed is \( v \cos \psi \) and the separation speed is \( v \sin \psi \). If the thickness of the zone is \( h \), then the average longitudinal strain rate perpendicular to the interface is \( (v \sin \psi)/h \) and the average shearing rate is \( (v \cos \psi)/h \), so that the principal strain rates are \( \frac{1}{2}(\pm 1 + \sin \psi)v/h \). In the standard material, then, the plastic dissipation per unit area of interface is \( cv \cos \phi \).

In another type of mechanism, introduced by Petterson and developed by Fellenius, the velocity discontinuities are along circular arcs, with the material inside an arc rotating rigidly about the center of the circle and thus sliding past the remaining material. This mechanism, known as a *slip circle*, is clearly appropriate only for a nondilatant (e.g., Tresca) material. Nevertheless, like the Coulomb mechanism with pure sliding, it is often used regardless of material properties. Both the location of the center of the circle and its radius can be chosen so as to minimize the upper-bound load predicted by the mechanism.

In a standard Mohr–Coulomb material, a velocity discontinuity such that the mass on one side rotates rigidly while that on the other side remains stationary takes the form of a logarithmic spiral rather than a circle. As can be seen from Figure 6.1.7, \( dr/(r \, d\theta) = \tan \phi \), which can be integrated to give

\[^1\text{In practice, soil mechanicians often use the Coulomb mechanism with pure sliding regardless of the yield criterion or flow rule assumed. Also, the mechanism is usually analyzed by means of statics rather than kinematics. For the Tresca material, the results are equivalent.}\]
\[ r = r_0 \exp[\tan \phi (\theta - \theta_0)]. \] Clearly, for \( \phi = 0 \), the curve becomes the circle \( r = r_0 \).

**Figure 6.1.7.** Velocity discontinuity in a standard Mohr–Coulomb material.

In accordance with Radenkovic’s theorems (see 3.5.1), any upper bound found for a standard Mohr–Coulomb material is also an upper bound for a nonstandard material with the same yield criterion, while any lower bound for a standard Mohr–Coulomb material whose angle of internal friction equals the dilatancy angle of the nonstandard material is also a lower bound for the latter. The following examples illustrate the procedure. The examples are limited to soils that can be modeled as homogeneous, a condition rarely encountered in real soil masses. For an extensive survey of applications of limit analysis to soil mechanics, including numerical results, see Chen [1975].

**Stability of a Vertical Bank**

A vertical bank of height \( h \) occupies the half-strip \( 0 \leq y \leq h, \ x \geq 0 \). We wish to determine the maximum height so that the bank does not collapse. Since the weight of the bank per unit horizontal area is \( wh \), where \( w \) is the specific weight, we may regard \( h \) as the equivalent of a load, and denote its greatest safe value by \( h_U \). A lower bound to \( h_U \) may be found by assuming, with Drucker and Prager [1952], the stress field \( \sigma_x = \tau_{xy} = 0 \), \( \sigma_y = -w(h - y) \), which satisfies the equilibrium equations and leaves the vertical and horizontal surfaces of the bank traction-free. The greatest numerical value of \( \sigma_y \), equal to \( wh \), must not exceed the yield stress in uniaxial compression for the standard Mohr–Coulomb material with internal-friction
angle $\psi$, namely
\[
\sigma_C' = \frac{2c \cos \psi}{1 - \sin \psi} = 2c \tan \left( \frac{\pi}{4} + \frac{\psi}{2} \right).
\]

If necessary — that is, if the bank is not situated atop a hard stratum — the stress field may be extended into the half-space $y \leq 0$ without violating the yield criterion as follows:
\[
\begin{align*}
\sigma_x &= \alpha wy, \\
\tau_{xy} &= 0, \\
\sigma_y &= \begin{cases} 
-w(h - y), & x > 0, \\
wy, & x < 0,
\end{cases}
\end{align*}
\]
where $\alpha = \frac{(1 - \sin \psi)}{(1 + \sin \psi)}$. The stress field thus contains admissible discontinuities that separate it into three zones.

For the standard Tresca material, the lower bound may be written as $2k/w$. An improvement to $2\sqrt{2}k/w$ is achieved by means of an admissible stress field consisting of seven zones, proposed by Heyman [1973], who also discusses incomplete stress fields leading to somewhat higher lower bounds. A numerical solution by Pastor [1976] yields a lower bound of $3.1k/w$.

However, the stress fields producing these improvements include tensile stresses in some regions, while the Drucker–Prager stress field does not. Indeed, for a material that cannot take tension it was shown by Drucker [1953], on the basis of a mechanism including a tension crack, that $2k/w$ is an upper bound as well.

![Figure 6.1.8. Coulomb mechanism in a vertical bank.](image)

An upper bound for a Mohr–Coulomb material may be found with the help of the Coulomb mechanism shown in Figure 6.1.8, in which a wedge of angle $\beta$ separates from the remainder of the bank. The weight (per unit thickness perpendicular to the page) of the wedge is $W = \frac{1}{2}wh^2 \tan \beta$, and if the magnitude of the velocity of the wedge is $v$, then its downward component is $v \cos(\beta + \phi)$. Equating the external work rate to the internal dissipation,
\[
\frac{1}{2}wh^2 \tan \beta \cos(\beta + \phi)v = chv \cos \phi \sec \beta, \quad (6.1.4)
\]
produces the upper bound
\[ h = \frac{2c \cos \phi}{w \sin \beta \cos(\beta + \phi)}. \]

The least upper bound is obtained by maximizing the denominator with respect to \( \beta \):
\[ \frac{d}{d\beta} [\sin \beta \cos(\beta + \phi)] = \cos \beta \cos(\beta + \phi) - \sin \beta \sin(\beta + \phi) = 0, \]
or
\[ \tan(\beta + \phi) = \cot \beta = \tan \left( \frac{\pi}{2} - \beta \right). \]

The equation is obeyed if \( \beta = \frac{1}{4}\pi - \frac{1}{2}\phi \). But
\[ \sin \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \cos \left( \frac{\pi}{4} + \frac{\phi}{2} \right) = \frac{1}{2}(1 - \sin \phi), \]
and therefore \( h_U \leq 2\sigma_C/w \) (= 4k/w for a Tresca material). We are thus left with the wide limits,
\[ \frac{2c}{w} \tan \left( \frac{\pi}{4} + \frac{\psi}{2} \right) \leq h_U \leq \frac{4c}{w} \tan \left( \frac{\pi}{4} + \frac{\phi}{2} \right). \]

The mechanism based on the logarithmic spiral produces a slight improvement over that of Coulomb: the optimal spiral reduces the factor in the upper bound from 4 to 3.83.

**Stability of a Simple Slope**

The improvement in the upper bound obtained by using a curved rather than a planar slip surface is significantly greater for inclined banks, known as simple slopes. It is conventional in soil mechanics to define the stability factor\(^1\) \( N_s = wh_U/c \), a function of the slope \( \alpha \) as well as of the internal-friction angle \( \phi \). The optimal Coulomb wedge gives
\[ N_s = \frac{4\sin \alpha \cos \phi}{1 - \cos(\alpha - \phi)}. \]

It was shown by Taylor [1937] that limit-equilibrium calculations based on the logarithmic spiral are so close to those based on the slip circle as to be undistinguishable. Chen [1975] showed further that there is no significant difference between the results of the limit-equilibrium method and upper-bound limit analysis. Some examples are shown in Table 6.1.1.

Very little work has to date been done on lower bounds in the limit analysis of slope stability.

\(^1\)Some authorities, including D. W. Taylor [1948], use the stability number \( m = 1/N_s \) instead.
Table 6.1.1. Stability Factors of Homogeneous Simple Slopes

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\phi$</th>
<th>Failure Surface</th>
</tr>
</thead>
<tbody>
<tr>
<td>$90^\circ$</td>
<td>0$^\circ$</td>
<td>Plane 4.00 Curved 3.83</td>
</tr>
<tr>
<td>25$^\circ$</td>
<td></td>
<td>6.28 6.03</td>
</tr>
<tr>
<td>$75^\circ$</td>
<td>0$^\circ$</td>
<td>Plane 5.21 Curved 4.56</td>
</tr>
<tr>
<td>25$^\circ$</td>
<td></td>
<td>9.80 6.03</td>
</tr>
<tr>
<td>$60^\circ$</td>
<td>0$^\circ$</td>
<td>Plane 6.93 Curved 5.24</td>
</tr>
<tr>
<td>25$^\circ$</td>
<td></td>
<td>17.36 12.74</td>
</tr>
</tbody>
</table>

Thrust on Retaining Walls

If a vertical bank of soil is too high for stability — and in a cohesionless soil, any height is too great — then it must be held back by a retaining wall. The soil and the wall exert on each other a mutual thrust, equal to the resultant of the horizontal stress $\sigma_x$ in the soil. Failure of the soil may occur when it is in one of two states, active and passive.

In a passive failure, the wall moves into the soil, increasing the horizontal pressure until the yield criterion is reached in the soil. The thrust $P$ is thus an increasing load on the soil mass, doing positive work, and its limiting value, known as the passive thrust and denoted $P_p$, is an ultimate load in the usual sense.

In an active failure, the wall is pushed outward as a result of the horizontal pressure, and this pressure is reduced until the soil yields. In the process the thrust decreases and does negative work. If the limiting thrust (the active thrust) is denoted $P_a$, then the upper and lower bounds are bounds on $-P_a$, and the usual nomenclature of limit analysis must be reversed: the upper-bound theorem gives a lower bound to $P_a$, and the lower-bound theorem gives an upper bound.

The static analysis is due to Rankine. The geometry is taken as the same as in the preceding problem, Figure 6.1.8, and the wall is assumed smooth. Equilibrium is satisfied if $\sigma_y = -w(h - y)$, $\tau_{xy} = 0$, and $\sigma_x$ depends on $y$ only. The Mohr–Coulomb criterion is assumed to be met everywhere, that is,

$$|\sigma_x - \sigma_y| + \sin \phi (\sigma_x + \sigma_y) = 2c \cos \phi.$$  

Two solutions exist for $\sigma_x$:

$$\sigma_x = \sigma_y \frac{1 - \sin \phi}{1 + \sin \phi} - 2c \frac{\cos \phi}{1 + \sin \phi}$$
$$= \sigma_y \tan^2 \left( \frac{\pi}{4} - \frac{\phi}{2} \right) - 2c \tan \left( \frac{\pi}{4} - \frac{\phi}{2} \right),$$
representing active failure, and

\[ \sigma_x = \sigma_y \frac{1 + \sin \phi}{1 - \sin \phi} + 2c \frac{\cos \phi}{1 - \sin \phi} \]
\[ = \sigma_y \tan^2 \left( \frac{\pi}{4} + \frac{\phi}{2} \right) + 2c \tan \left( \frac{\pi}{4} + \frac{\phi}{2} \right), \]

representing passive failure. Introducing \( \sigma_y = -w(h - y) \) and integrating over \( 0 \leq h \) gives the limiting thrusts,

\[ P_a = \frac{1}{2} wh^2 \tan^2 \left( \frac{\pi}{4} - \frac{\phi}{2} \right) - 2ch \tan \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \]

and

\[ P_p = \frac{1}{2} wh^2 \tan^2 \left( \frac{\pi}{4} + \frac{\phi}{2} \right) + 2ch \tan \left( \frac{\pi}{4} + \frac{\phi}{2} \right). \]

The preceding formulas, known as Rankine’s formulas, are widely used in soil mechanics. In view of Radenkovic’s second theorem, however, it must be recognized that they are not true lower bounds (in the usual sense for \( P_p \), in the reverse sense for \( P_a \)) unless the friction angle \( \phi \) is replaced by the dilatancy angle \( \psi \). Upper bounds can be obtained by means of the Coulomb mechanism, following Coulomb’s own analysis of 1776, but making sure that a kinematic approach with an associated flow rule is taken; Coulomb assumed pure sliding, and analyzed the wedge statically.

For active failure, the mechanism is the same as for the free-standing vertical bank. With the wall again assumed to be smooth, the external rate of work is given by the left-hand side of (6.1.4) with the additional term \(-Pv \sin(\beta + \phi)\), and the internal dissipation equals the right-hand side. Consequently,

\[ P = \frac{1}{2} wh^2 \tan \beta \cot(\beta + \phi) - ch \cos \phi \sec \beta \csc(\beta + \phi). \]

Both terms on the right-hand side can be shown to be stationary at \( \beta = \frac{1}{4} \pi - \frac{1}{4} \phi \), so that the largest \( P \) (corresponding to the smallest upper bound on \(-P\)) is given by

\[ P_a = \frac{1}{2} wh^2 \tan^2 \left( \frac{\pi}{4} - \frac{\phi}{2} \right) - ch \cos \phi \sec^2 \left( \frac{\pi}{4} - \frac{\phi}{2} \right). \]

But

\[ \cos \phi = \sin \left( \frac{\pi}{2} - \phi \right) = \sin 2 \left( \frac{\pi}{4} - \frac{\phi}{2} \right) = 2 \sin \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \cos \left( \frac{\pi}{4} - \frac{\phi}{2} \right), \]

so that

\[ \cos \phi \sec^2 \left( \frac{\pi}{4} - \frac{\phi}{2} \right) = 2 \tan \left( \frac{\pi}{4} - \frac{\phi}{2} \right), \]
and the Rankine formula for the active thrust is recovered.

An analogous result is obtained for the passive thrust. Here the wedge moves upward and to the right, the velocity forming an angle $\beta - \phi$ with the vertical (details are left to an exercise). We thus see that the Rankine formulas give the correct limiting thrusts on a smooth wall for the standard Mohr–Coulomb material. For the nonstandard material, they furnish the Radenkovic bounds.

The kinematic approach may be extended to obtain upper bounds in the presence of friction between the soil and the wall, by adding the term $\mu P|v_y|$ to the internal dissipation, $\mu$ being an average coefficient of friction and $v_y$ the vertical component of velocity; it is assumed that the wall moves horizontally. While no analytical solutions exist, the wedge angle $\beta$ giving the lowest upper bound can easily be found numerically for given $\mu$, $\phi$, and $wh/c$.

**Exercises: Section 6.1**

1. Find the best value of the wedge angle $\gamma$ for the stress field of Figure 6.1.1(a).

2. Assume that the velocity discontinuity in Figure 6.1.1(b) is inclined at an arbitrary angle $\alpha$. Find the upper bound to the cutout factor, and show that $\alpha = 45^\circ$ gives the best upper bound.

3. Show that for a rectangular slab of sides $2a$ and $2b$ ($b > a$) with a slit of length $2\beta a$ parallel to the shorter side, the cutout factor for simple tension perpendicular to the slit is still $1 - \beta$.

4. Find lower and upper bounds on the cutout factor for the slab in Exercise 1 when $b < a$.

5. Find the value of the load $F$ corresponding to the slip-line fields of Figure 6.1.4(a) for $\theta \geq 75^\circ$.

6. Determine the velocity fields corresponding to the mechanisms of Figure 6.1.5(b) and (c) and the corresponding upper bounds to the collapse load $F$.

7. Derive Equation (6.1.2) for beams with $h/L \leq \tau_Y/\sigma_Y$.

8. Derive Equation (6.1.3) for beams with $h/L \geq \tau_Y/\sigma_Y$.

9. Find an upper bound to the critical height $h$ of the bank of Figure 6.1.8 when the straight velocity-discontinuity line is replaced by a logarithmic spiral (Figure 6.1.7).
10. Derive the equations governing the upper bound to the passive thrust and the lower bound to the active thrust on a retaining wall in the presence of friction between the soil and the wall.

11. Assuming plane strain, use a stress field like that of Figure 5.1.5(e) to find a lower bound to the ultimate tensile force carried by the symmetrically notched tension specimen of Exercise 9, Section 5.1. Compare with the result of that exercise.

Section 6.2 Beams Under Combined Stresses

6.2.1. Generalized Stress

Introduction

A concept of great usefulness in the limit analysis of beams, arches, frames, plates and shells was introduced by Prager [1955b, 1956b, 1959]. It is that of generalized stress and strain.

Consider the ideal sandwich beam as shown in Figure 3.5.1(a) (page 155), but subject to distributed loading so that the axial force $P$ and bending moment $M$ vary along its length. While $M$ and $P$ can no longer be regarded as generalized loads, they may be regarded as generalized stresses in the following sense. At any section of the beam, the stresses in the flanges are $\sigma = \frac{P}{2A} \pm \frac{M}{Ah}$. The local values of the elongation and rotation obey the relations

$$\frac{d\Delta}{dx} = \varepsilon, \quad \frac{d\theta}{dx} = \kappa,$$

where $\varepsilon$ is the mean longitudinal strain and $\kappa$ is the curvature. The strains in the flanges are $\varepsilon \pm h\kappa/2$, and the internal virtual work can easily be shown to be

$$\delta W_{int} = \int_0^L (P \delta \varepsilon + M \delta \kappa) dx,$$

the span of the beam being $0 < x < L$. The local axial force $P$ and bending moment $M$ may now be regarded as the generalized stresses, with $\varepsilon$ and $\kappa$, respectively, as their conjugate generalized strains.

Figure 3.5.1(a), in addition to representing the limit-load locus for the beam under external axial force and moment, thus also represents the yield locus for the ideal sandwich beam. Such yield loci are also called interaction diagrams; an example was already studied in 4.4.1 (see Figure 4.4.5(a), page 224).

The ideal sandwich beam is statically determinate in the sense of 4.1.1, since it has no range of contained plastic deformation: if the material is
perfectly plastic, then the beam can undergo unlimited deformation as soon
as either flange yields. In any real beam, as we already know, the ultimate
moment $M_U$ is greater than the elastic-limit moment $M_E$, and therefore
two distinct yield loci exist: the elastic-limit (or initial yield) locus and the
ultimate yield locus [again, see Figure 4.4.5(a)]. Under the hypothesis of
rigid–plastic behavior, however, only the ultimate yield locus is relevant.

**Generalized Stress and Strain: Definitions**

Generalized stresses may coincide with the actual stresses, or they may
be local stress resultants integrated over one or (as in the present example)
two dimensions, or even over a whole finite element of the body (such as a
bar in a truss). If the generalized stresses are denoted $Q_j$ ($j = 1, \ldots, n$),
then the conjugate generalized strains $q_j$ are in general defined by

$$
\delta W_{int} = \int_{\Omega} \sum_{j=1}^{n} Q_j \delta q_j d\Omega,
$$

where $\int_{\Omega} (\cdot) d\Omega$ describes integration over the entire body with respect to vol-
ume, area, or length, as appropriate, or summation over all finite elements.¹

Let $Q$ and $q$ denote the generalized stress and strain vectors, respec-
tively. As illustrated by the ideal sandwich beam, a yield locus in terms of
generalized stresses, say $\Phi(Q) = 0$, may be derived in exactly the same way
as the limit locus in terms of generalized loads was derived in 3.5.1. For
rigid–plastic materials, the generalized plastic dissipation is thus

$$
\bar{D}_p = Q \cdot \dot{q},
$$

and the principle of maximum plastic dissipation may be written as

$$(Q - Q^*) \cdot \dot{q} \geq 0$$

or

$$
\bar{D}_p(\dot{q}) \geq Q^* \cdot \dot{q}
$$

(6.2.2)

for any $Q^*$ such that $\Phi(Q^*) \leq 0$.

Finally, the theorems of limit analysis may be restated as follows.

**Lower-Bound Theorem.** A load point $P$ is on or inside the limit locus if
a generalized stress field $Q^*$ can be found that is in equilibrium with $P$ and
obeys $\Phi(Q^*) \leq 0$ everywhere.

**Upper-Bound Theorem.** A load point $P$ is on or outside the limit locus if a kinematically admissible velocity field, yielding the generalized velocity

¹In technical mathematical language, $d\Omega$ is a measure in a space of three, two, one or
zero dimensions.
vector $\mathbf{p}^*$ conjugate to $\mathbf{P}$ and the generalized strain-rate field $\mathbf{q}^*$, can be found so that

$$\mathbf{P} \cdot \mathbf{p}^* = \int_{\Omega} \bar{D}_p(\mathbf{q}^*) \, d\Omega.$$  \hspace{1cm} (6.2.3)

**Elastic and Plastic Generalized Strain**

When it is desired to describe elastic–plastic behavior in terms of generalized stress and strain, then it is necessary to decompose the generalized strain into elastic and plastic parts:

$$\mathbf{q} = \mathbf{q}^e + \mathbf{q}^p.$$  

But with the exception of some simple cases, there is in general no one-to-one correspondence between $\mathbf{q}^e$ and $\mathbf{\varepsilon}^e$ or between $\mathbf{q}^p$ and $\mathbf{\varepsilon}^p$. Consider, for example, a real (as distinct from ideal) beam subject to symmetric bending only; the moment $M$ is the only generalized stress, and the curvature $\kappa$ is the only generalized strain. The actual strain at a point is given by

$$\mathbf{\varepsilon} = -\kappa y.$$  

In the elastic range, the moment-curvature relation is $M = EI\kappa$, and therefore the elastic part of the curvature is

$$\kappa^e = \frac{M}{EI}.$$  

The plastic strain is

$$\varepsilon^p = \mathbf{\varepsilon} - \mathbf{\varepsilon}^e = -(\kappa^e + \kappa^p)y - \frac{\sigma}{E}$$

$$= -\kappa^p y - \frac{1}{E} \left( \sigma + \frac{My}{I} \right).$$

Thus, while

$$\kappa^p = -\frac{1}{I} \int_A y \varepsilon^p \, dA,$$

there is no inverse relation by which $\kappa^p$ determines $\varepsilon^p$. The quantity $\sigma + My/I$ does not vanish in the range of contained plastic deformation, and neither does its time derivative. Consequently,

$$\int_A \sigma \dot{\varepsilon}^p \, dA \neq M \dot{\kappa}^p,$$

except when $\dot{\sigma} = 0$, a condition that implies that $\dot{M} = 0$ and hence $\dot{\kappa}^e = 0$, and therefore holds only on the ultimate yield locus.

It follows that the principle of maximum plastic dissipation in terms of generalized stress and generalized plastic strain is in general valid only under unrestricted plastic flow, with $\dot{\mathbf{q}}^p = \mathbf{\dot{q}}$. The exceptional cases are those in which no contained plastic deformation occurs *locally*, as at a section of an ideal beam, or in a bar carrying axial force only (truss member).
6.2.2. Extension and Bending

Introduction

The theory of symmetric pure bending of elastic–plastic beams, discussed in 4.4.1, can easily be extended to beams subject to an axial force $P$ in addition to the bending moment $M$, provided the deflection is so small that the additional bending moment resulting from the axial force acting over the deflection (the so-called $P$-$\Delta$ effect) is negligible. With this proviso satisfied, and with the uniaxial stress-strain relation given by $\sigma = f(\varepsilon)$, the first of Equations (4.4.7) needs only to have its right-hand side changed from 0 to $P$.

As seen in 4.4.1, the problem of asymmetric bending is difficult even in the absence of axial force. An analysis of rectangular beams subject to axial force and bending moments about both axes was carried out by Shakir-Khalil and Tadros [1973]. In the present subsection, only symmetric bending is considered.

Consider a beam whose centroidal fiber coincides with the $z$-axis, whose cross-section is symmetric about the $y$-axis, and which is made of an elastic–perfectly plastic material. In bending in the $yz$-plane, the strain is given as

$$\varepsilon = -\kappa (y - y_0),$$

so that $\kappa y_0 = \varepsilon_0$ is the strain of the centroidal fibers or, equivalently, the average strain over the cross-section.

In the elastic range, this strain is elastically related to the average stress $P/A$, that is, $\kappa y_0 = P/AE$. Since $\kappa = M/EI$ regardless of the value of $P$, it follows that the coordinate $y_0$ of the neutral axis is given by

$$y_0 = \frac{PI}{MA},$$

and the stress distribution is

$$\sigma = \frac{P}{A} - \frac{My}{I}.$$

Let the $y$-coordinates of the extreme bottom and top fibers be $y = -c_1$ and $y = c_2$, respectively. The magnitudes of the extreme values of the stresses are

$$\sigma_1 = \left| \frac{P}{A} + \frac{Mc_1}{I} \right|, \quad \sigma_2 = \left| \frac{P}{A} - \frac{Mc_2}{I} \right|,$$

and the elastic limit corresponds to $\max(\sigma_1, \sigma_2) = \sigma_Y$, that is,

$$\max \left( \left| \frac{P}{P_U} + \frac{M}{M_{E1}} \right|, \left| \frac{M}{M_{E2}} - \frac{P}{P_U} \right| \right) = 1,$$

where

$$P_U = \sigma_Y A, \quad M_{E1} = \sigma_Y \frac{I}{c_1}, \quad M_{E2} = \sigma_Y \frac{I}{c_2}.$$

If the section is doubly symmetric, then $c_1 = c_2 = h/2$ and $M_{E1} = M_{E2} = M_E = 2\sigma_Y I/h$. The elastic-limit locus is then simply $|P/P_U| + \left| \frac{My}{I} \right|$.
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Figure 6.2.1. Elastic-limit locus for an asymmetric beam under combined bending and extension.

\[ |\frac{M}{M_E}| = 1, \text{ that is, it has exactly the same form as the yield locus of the ideal sandwich beam [Figure 3.5.1(b)] but with the elastic-limit moment } M_E \text{ replacing the ultimate moment } M_U. \]

For a section without double symmetry, the elastic-limit locus is as shown in Figure 6.2.1 if \( c_2 > c_1 \) (so that \( M_{E1} > M_{E2} \)).

Note that for a certain ratio of \( P \) to \( M \), namely \( \frac{P}{M} = \frac{(c_2 - c_1)A}{2I} \), yielding occurs simultaneously at the top and bottom fibers, and two plastic zones form as the generalized stress \((M, P)\) moves outside the elastic-limit locus. Otherwise only one plastic zone forms at first. As the point \((M, P)\) moves farther from the origin, the stress at the extreme fiber opposite the plastic zone increases until it, too, reaches the yield-stress value, creating a second plastic zone. Further loading results in the shrinking of the elastic core, exactly as in pure bending, until it becomes negligibly thin, indicating that the ultimate yield locus has been reached.

Yield Locus for Symmetric Bending

In order to determine the ultimate yield locus, let us assume for the sake of definiteness that \( M > 0 \), so that the bottom fibers are in tension, with \( \sigma = \sigma_Y \), and the top fibers are in compression, with \( \sigma = -\sigma_Y \). If \( y_0 \) is the

\[ ^1 \text{For the ideal sandwich beam, of course, } M_E = M_U. \]
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$y$-coordinate of the vanishing elastic core, then

$$M = -\sigma_Y \left[ \int_{y=-c_1}^{y=y_0} y \, dA - \int_{y=y_0}^{y=c_2} y \, dA \right] = 2\sigma_Y \int_{y=y_0}^{y=c_2} y \, dA \quad (6.2.4)$$

and

$$P = \sigma_Y \left[ \int_{y=-c_1}^{y=y_0} dA - \int_{y=y_0}^{y=c_2} dA \right] \quad (6.2.5)$$

since $\int_{y=-c_1}^{y=c_2} y \, dA = 0$. For a given cross-section, the integrals in Equations (6.2.4)–(6.2.5) can be evaluated in terms of $y_0$, so that these equations furnish a parametric representation of the ultimate yield locus in the $MP$-plane in terms of the parameter $y_0$, the range of $y_0$ being $-c_1 \leq y_0 \leq c_2$.

**Associated Flow Rule**

The principle of maximum plastic dissipation holds in states of uniaxial stress regardless of the yield criterion and flow rule obeyed by the material under multiaxial stress. Consequently the generalized strain rate $(\dot{\varepsilon}_0, \dot{\kappa})$ must be perpendicular to the yield locus in the $MP$-plane, that is,

$$\dot{\varepsilon}_0 \, dP + \dot{\theta} \, dM = 0.$$

Now consider Equations (6.2.4)–(6.2.5). If $b(y)$ denotes the width of the beam at the level $y$, then $dA = b(y) \, dy$, and therefore

$$dM = -2\sigma_Y b(y_0) y_0 \, dy_0, \quad dP = 2\sigma_Y b(y_0) \, dy_0,$$

so that normality is equivalent to $\dot{\varepsilon}_0 - y_0 \dot{\kappa} = 0$ — precisely the definition of $\dot{\varepsilon}_0$.

**Examples of Yield Loci**

In order to compare the behavior of different cross-sections, it is convenient to describe the yield locus in terms of the dimensionless generalized stresses $m = M/M_U$ and $p = P/P_U$, where $P_U = \sigma_Y A$ ($A$ being the total cross-sectional area) and $M_U$ is given by Equation (4.4.11). The parameter can also be made dimensionless by defining, for example, $\eta = 2y_0/h$, where $h = c_1 + c_2$ is the depth of the beam; the range of $\eta$ is thus $-\eta_1 \leq \eta \leq \eta_2$, where $\eta_i = 2c_i/h$, $i = 1, 2$. Equations (6.2.4)–(6.2.5) can therefore be written symbolically as

$$m = \bar{m}(\eta), \quad p = \bar{p}(\eta). \quad (6.2.6)$$

As an example, consider a rectangular beam of width $b$ and depth $h$. It is easy to see that Equations (6.2.4)–(6.2.5) become

$$M = \sigma_Y b \left( \frac{h^2}{4} - y_0^2 \right)$$
and

\[ P = 2\sigma_Y b y_0 \]

In dimensionless form,

\[ m = 1 - \eta^2, \quad p = \eta, \]

so that in this case the yield locus can be described in explicit form,

\[ m = 1 - p^2, \]

and forms a parabola.

For a circular bar of radius \( a \), it is more convenient to define \( \eta \) by \( \eta = \sin^{-1}(y_0/a) \), so that its range is \(-\frac{1}{2}\pi \leq \eta \leq \frac{1}{2}\pi\). The yield locus is given by

\[ p = \frac{1}{\pi}(2\eta + \sin 2\eta), \quad m = \cos^3 \eta. \]

If \( M \) is negative, it need only be replaced by \(-M\) (and \( m \) by \(-m\)) in all the results for doubly symmetric sections. Without double symmetry, \( P \) must also be replaced by \(-P\) (and \( p \) by \(-p\)).

The yield loci for the rectangular and circular bars are shown as curves 1 and 2 in Figure 6.2.2. It is seen that they differ only slightly over the entire range. The yield locus for an I-beam, on the other hand, can be expected to lie between the loci for the rectangular beam and for the ideal sandwich beam, and closer to the latter. Figure 6.2.2 also shows, as curve 3, the locus for the I-beam shown in the adjacent picture.

In practice, curved yield loci such as the one for the rectangular bar are often replaced, following Onat and Prager [1954], by piecewise linear approximations such as the one shown with dashed lines in Figure 6.2.2. Such approximations may also be regarded as the exact yield loci for certain idealized sections, which are in turn approximations to the true sections. The advantage of this point of view (see Hodge [1959], Section 7-3) is that a velocity field that gives generalized strain rates associated with the approximate yield locus may be easily visualized in the context of the idealized section. A frequently used yield locus for wide-flange steel sections is given by

\[ 0.85|m| + |p| = 1, \quad |p| \geq 0.15, \]

\[ |m| = 1, \quad |p| \leq 0.15. \]

(6.2.7)

**Application: Collapse of a Semicircular Arch**

As an illustration of the use of the yield locus in terms of moment and axial force, we consider the pinned-ended semicircular arch of radius \( a \), loaded by a concentrated vertical force \( 2F \) at midspan, shown in Figure 6.2.3(a).
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Figure 6.2.2. Yield loci under combined bending and extension for rectangular beams (curve 1), circular beams (curve 2), and the I-beam shown (curve 3). Dashed lines represent a piecewise linear approximation for rectangular beams.

The vertical reactions at the supports are each equal to $F$ by symmetry, but the horizontal reactions $\pm H$ are unknown. Let $\chi$ denote the angle between the resultant reaction and the vertical, so that $\tan \chi = H/F$. A free-body diagram of a segment of the arch is shown in Figure 6.2.3(b), and equilibrium shows that the axial force and moment are

$$P(\phi) = -F \sin \phi - H \cos \phi = -F(\sin \phi + \tan \chi \sin \phi)$$

$$= -\frac{F}{\cos \chi} \sin(\chi + \phi),$$

$$M(\phi) = Fa(1 - \sin \phi) - Ha \cos \phi$$

$$= \frac{Fa}{\cos \chi} [\cos \chi - \sin(\chi + \phi)].$$

With the moment and axial force varying along the arch, it must be assumed that the yield criterion (whichever is chosen) is met only at certain critical sections. As in the case of transversely loaded beams discussed in 4.4.2, plastic hinges form at those sections. If a single hinge were to form in the arch, it would necessarily, because of symmetry, be at midspan. A three-hinged arch, however, is a stable structure. Consequently, plastic collapse of the arch requires the formation of two additional plastic hinges, located
symmetrically about the midpoint of the arch. The collapse mechanism is shown by the dashed curves in Figure 6.2.3(a); the arch segments between the hinges move as rigid bodies.

![Figure 6.2.3. Pin-ended circular arch: (a) initial geometry and loading (solid line) and collapse mechanism (dashed line); (b) free-body diagram of a segment.](image)

The location of the off-center hinges can easily be determined by noting that both moment and axial force have local maxima only at $\phi = \pi/2 - \chi$, and therefore any convex combination of them also has a local maximum there. Collapse is thus determined by the requirement that the yield criterion $\Phi(M, P) = 0$ is met at $\phi = 0$ and $\phi = \pi/2 - \chi$. Assuming a section with double symmetry so that $\Phi(M, P) = \Phi(|M|, |P|)$, we can find $\chi$ and the value of $F$ at collapse by solving simultaneously

$$\Phi(F \tan \chi, Fa|1 - \tan \chi|) = 0 \quad \text{and} \quad \Phi(F \sec \chi, Fa(\sec \chi - 1)) = 0.$$  

The choice of a nonlinear yield criterion necessitates the simultaneous solution of two nonlinear equations, a task that may be unpleasant. With a piecewise linear yield criterion, on the other hand, once the side of the polygon is chosen that is appropriate for each equation, the equations are linear in $F$. Eliminating $F$, a quadratic equation in $\cos \chi$ or $\sin \chi$ is obtained.

For simplicity, the square yield locus corresponding to the ideal sandwich section will be used. With the help of the dimensionless quantity $\eta = M_U/P_Ua$, the equations to be solved are

$$|1 - \tan \chi| + \eta \tan \chi|Fa = M_U,$$

$$|(1 + \eta) \sec \chi - 1|Fa = M_U.$$  

Consequently $\chi$ satisfies

$$|1 - \tan \chi| + \eta \tan \chi = (1 + \eta) \sec \chi - 1.$$  

It can immediately be seen that if $\tan \chi > 1$, then $\tan \chi = \sec \chi$, or $\chi = \pi/2$, so that the additional plastic hinges coincide with the central
hinge, and no collapse is possible. It is necessary, then, that \( \tan \chi < 1 \). For \( \eta \) negligibly small, \( \tan \chi \) is very nearly \( 3/4 \) and therefore \( F \) is a result equivalent to neglecting the influence of axial forces. For \( \eta \) small but not negligible, a first-order approximation in \( \eta \) may be effected, leading to \( \tan \theta = 3/4 - 5\eta/16 \) and \( F = (4M/a)/(1 + 4.25\eta) \). Other examples of arch collapse based on piecewise linear yield loci were studied by Onat and Prager [1953, 1954]; see also Hodge [1959], Section 7-4.

### 6.2.3. Combined Extension, Bending and Torsion

In a bar under a combination of axial force, bending moment, and torque, the nonvanishing stress components are \( \sigma_x \), \( \tau_{xz} \), and \( \tau_{yz} \), with \( \tau_{xz} = \partial \phi / \partial y \) and \( \tau_{yz} = -\partial \phi / \partial x \), \( \phi \) being the stress function. Any point of the bar is in a state of plane stress in the plane that is perpendicular to \( \nabla \phi \); the state of stress in that plane can be given by

\[
\begin{bmatrix}
\sigma \\
\tau \\
0
\end{bmatrix}
\]

where \( \sigma = \sigma_z \) and \( \tau = |\nabla \phi| = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} \). Both the Mises and the Tresca yield criteria are given by Equation (3.3.5):

\[
\left( \frac{\sigma}{\sigma_Y} \right)^2 + \left( \frac{\tau}{\tau_Y} \right)^2 = 1.
\]

Since \( \sigma_{ij}\dot{\varepsilon}_{ij} = \sigma \dot{\varepsilon} + \tau \dot{\gamma} \), the associated flow rule may be obtained from the maximum-plastic-dissipation principle as

\[
\dot{\varepsilon}^p = \dot{\lambda} \frac{\sigma}{\sigma_Y}, \quad \dot{\gamma}^p = \dot{\lambda} \frac{\tau}{\tau_Y},
\]

and the plastic dissipation is

\[
D_p(\dot{\varepsilon}) = \sqrt{(\sigma_Y \dot{\varepsilon})^2 + (\tau_Y \dot{\gamma})^2}.
\]

#### Lower Bound

A lower-bound yield locus for combined extension, bending, and torsion can be found by generalizing an approach proposed by Hill and Siebel [1951] for combined bending and torsion only. The approach is based on assuming, on the one hand, the same distribution of normal stress as in 6.2.2, but with \( |\sigma| = \alpha \sigma_Y \), where \( 0 \leq \alpha \leq 1 \); and on the other hand, the same distribution of shear stress as in fully plastic torsion, but with \( \tau = \sqrt{1 - \alpha^2} \tau_Y \). The second assumption leads immediately to

\[
t = \frac{T}{T_U} = \sqrt{1 - \alpha^2}.
\]

The first assumption means that the \( m-p \) relation would be given by (6.2.6) if \( m \) and \( p \) were defined as \( M/\alpha MU \) and \( P/\alpha PU \), respectively. With the
standard definitions, the relation is therefore

\[ \frac{p}{\alpha} = \tilde{p}(\eta), \quad \frac{m}{\alpha} = \tilde{m}(\eta). \]

Eliminating \( \alpha \), we can describe the yield surface in \( mpt \)-space in terms of the single parameter \( \eta \):

\[ p = \sqrt{1 - t^2} \tilde{p}(\eta), \quad m = \sqrt{1 - t^2} \tilde{m}(\eta). \]

For the rectangular beam, we have the explicit description

\[ m = \sqrt{1 - t^2} - \frac{p^2}{\sqrt{1 - t^2}}. \]

The projections of this surface on both the \( mt \)- and \( pt \)-planes are unit circles.

**Upper Bound**

An upper-bound yield curve for combined extension, bending and torsion can be found following the method of Hill and Siebel [1953]. The assumed velocity field is taken so that it results only in an extension rate \( \dot{\varepsilon}_0 \) of the centroidal fiber, a pure curvature rate \( \dot{\kappa} \) about the centroidal axis, and a rate of twist \( \dot{\theta} \) about the centroid; warping is neglected.\(^1\) With symmetric bending in the \( yz \)-plane assumed, the strain rates are

\[ \dot{\varepsilon}_z = \dot{\varepsilon}_0 - \dot{\kappa} y, \quad \dot{\gamma}_{\theta z} = r \dot{\theta}, \]

where \( r = \sqrt{x^2 + y^2} \). The plastic dissipation is therefore

\[ D_p(\dot{\varepsilon}^*) = \tau_Y |\dot{\theta}| \sqrt{r^2 + (\alpha y - \beta)^2}, \]

where \( \alpha = (\sigma_Y/\tau_Y)(\dot{\kappa}/\dot{\theta}) \) and \( \beta = (\sigma_Y/\tau_Y)(\dot{\varepsilon}_0/\dot{\theta}) \).

A simple way of finding the upper-bound generalized stresses corresponding to the assumed velocity field is to determine the stresses that are related to it by the associated flow rule and that obey the yield criterion — but that do not, in general, form a statically admissible stress field — and then to calculate their resultants. The associated flow rule (6.2.8) produces the stresses

\[ \sigma = \frac{\sigma_Y^2}{\lambda} (\dot{\varepsilon}_0 - \dot{\kappa} y) = -\mu \sigma_Y (\alpha y - \beta) \]

and

\[ \tau = \mu \tau_Y r, \]

\(^1\)A correction for warping, resulting in better upper bounds for noncircular sections, is due to Gaydon and Nuttall [1957].
where \( \tau = \tau_{\theta z} \) (so that \( \tau_{rz} = 0 \)),\(^1\) and \( \mu = \tau_{\gamma} \dot{\theta} / \dot{\lambda} \) is a function of position that can be determined by requiring that the yield criterion (3.3.5) be obeyed everywhere, resulting in

\[
\mu = \frac{1}{\sqrt{r^2 + (\alpha y - \beta)^2}}.
\]

The stress resultants \( M, P \) and \( T \) are therefore given by

\[
M = -\int_A y\sigma dA = \alpha \sigma Y \int_A \frac{y^2}{\sqrt{r^2 + (\alpha y - \beta)^2}} dA,
\]
\[
P = \int_A \sigma dA = \beta \sigma Y \int_A \frac{1}{\sqrt{r^2 + (\alpha y - \beta)^2}} dA,
\]
\[
T = \int_A r\tau dA = \tau_{\gamma} \int_A \frac{r^2}{\sqrt{r^2 + (\alpha y - \beta)^2}} dA.
\]

The integrations in Equations (6.2.10) may be performed, numerically if necessary, to yield a parametric representation of the upper-bound yield surface in \( mpt \)-space in terms of the parameters \( \alpha \) and \( \beta \). Computed yield curves representing the projections of the lower-bound and upper-bound yield surfaces in the \( mt \)- and \( pt \)-planes, are shown in Figure 6.2.4(a) and (b) for a circular and a square bar, respectively.

**Extension and Torsion of a Circular Bar**

For a circular bar, the upper-bound solution presented above is, in fact, a complete solution, since an axisymmetric shear-stress distribution \( \tau_{\theta z} = \tau_\gamma \), \( \tau_{rz} = 0 \) is statically admissible. A closed-form result can be obtained for extension and torsion alone, that is, for \( \alpha = 0 \).

We define the dimensionless parameter \( \zeta = \beta / a \), where \( a \) is the radius of the bar. Equations (6.2.10)\(^2\),\(^3\) give

\[
p = \frac{2\pi}{\sigma_Y \pi a^2} \zeta \sigma Y \int_0^a \frac{r \, dr}{\sqrt{\zeta^2 a^2 + r^2}} = \zeta \int_0^a \frac{dx}{\sqrt{x + \zeta^2}}
\]
and

\[
t = \frac{2\pi}{\pi a^3 \gamma Y} \int_0^a \frac{r^3 \, dr}{\sqrt{\zeta^2 a^2 + r^2}} = \frac{3}{2} \int_0^1 \frac{x \, dx}{\sqrt{x + \zeta^2}},
\]
or

\[
p = 2\zeta \left( \sqrt{\zeta^2 + 1} - \zeta \right), \quad t = 2\zeta^3 - (2\zeta^2 - 1)\sqrt{\zeta^2 + 1}.
\]

The first of Equations (6.2.11) can be rewritten as

\[
p + 2\zeta^2 = 2\zeta \sqrt{\zeta^2 + 1}.
\]

\(^1\)Thus the traction boundary conditions are not satisfied for any but a circular bar.
Squaring both sides, we find
\[ 4\zeta^4 + 4\zeta^2 = 4\zeta^4 + 4\zeta^2 p + p^2, \]
or \[ 4\zeta^2(1 - p) = p^2, \] which can immediately be solved for \( \zeta \):
\[ \zeta = \frac{p}{2\sqrt{1 - p}}. \]
Substituting in the equation for \( t \), we obtain the explicit relation
\[ t = \frac{1}{2}(2 + p)\sqrt{1 - p}. \]
Squaring both sides of this equation, we may rewrite it as
\[ t^2 + p^2 = 1 + \frac{p^2 - p^3}{4}, \]
a form that is convenient for determining the extent to which the present upper-bound yield curve differs from the previously found lower bound, described by the unit circle. The right-hand side of the last equation has its
maximum at \( p = 2/3 \), its value there being 28/27. Consequently the distance of the points on the dimensionless yield curve from the origin lies between 1 and \( \sqrt{28/27} = 1.018 \), and the curve differs only slightly from the circle.

The stress distribution giving rise to the resultants just obtained can also be determined as the solution of a constrained extremum problem: find the stresses satisfying (3.3.5) everywhere such that the torque,

\[ T = \int_A r \tau \, dA, \]

is maximum for a given axial force,

\[ P = \int_A \sigma \, dA. \]

Substituting for \( \tau \) by solving (3.3.5), the problem may be written as

\[ \tau Y \delta \int_A r \sqrt{1 - \left(\frac{\sigma}{\sigma_Y}\right)^2} \, dA + \nu \delta \int_A \sigma \, dA = 0, \]

where \( \nu \) is a Lagrangian multiplier, or

\[ \int_A \left[ \beta - \frac{\left(\frac{\sigma}{\sigma_Y}\right) r}{\sqrt{1 - \left(\frac{\sigma}{\sigma_Y}\right)^2}} \right] \delta \sigma dA = 0, \]

where \( \beta = \nu \sigma_Y / \tau_Y \). There being no further constraint on the distribution of \( \sigma \), the quantity in brackets under the integral sign must be zero, that is,

\[ \frac{\sigma}{\sigma_Y} = \frac{\beta}{\sqrt{\beta^2 + r^2}}, \]

and the yield criterion is satisfied everywhere if and only if

\[ \frac{\tau}{\tau_Y} = \frac{r}{\sqrt{\beta^2 + r^2}}. \]

Integration leads immediately to \( p \) and \( t \) given by Equations (6.2.11), with \( \zeta = \beta/a \) as before.

**Bending and Torsion of a Bar of Arbitrary Cross-Section**

A complete solution for combined bending and torsion of a fully plastic bar of arbitrary cross-section may be obtained by integrating numerically a nonlinear partial differential equation first derived by Handelman [1944]. The stress function \( \phi(x, y) \) is now regarded as unknown, and it is to be found so that it maximizes the bending moment for a given torque (or vice versa). For doubly symmetric bending in the \( yz \)-plane, we have

\[ M = \frac{\sigma_Y}{\tau_Y} \int_A |y| \sqrt{\tau_Y^2 - \left(\partial \phi / \partial x\right)^2 - \left(\partial \phi / \partial y\right)^2} \, dA. \]
The constrained extremum problem may therefore be written as

\[
\int_A \left[ y \sqrt{\frac{\tau_Y^2}{\tau_Y^2} - (\frac{\partial \phi}{\partial x})^2 - (\frac{\partial \phi}{\partial y})^2} - \nu \delta \phi \right] dA = 0,
\]

where \( \nu \) is again a Lagrangian multiplier. Since \( \phi = 0 \) and hence \( \delta \phi = 0 \) on the boundary, integration by parts leads to

\[
\int_A \left\{ \frac{\partial}{\partial x} \left[ \frac{y \partial \phi / \partial x}{\sqrt{\tau_Y^2 - (\frac{\partial \phi}{\partial x})^2 - (\frac{\partial \phi}{\partial y})^2}} \right] + \frac{\partial}{\partial y} \left[ \frac{y \partial \phi / \partial y}{\sqrt{\tau_Y^2 - (\frac{\partial \phi}{\partial x})^2 - (\frac{\partial \phi}{\partial y})^2}} \right] + \nu \right\} \delta \phi dA = 0.
\]

Since \( \phi \) is unconstrained in the interior of \( A \), the contents of the curly brackets must vanish everywhere, and this furnishes the required partial differential equation for \( \phi \), an elliptic equation subject to the boundary condition \( \phi = 0 \). Different values of \( \nu \) give different ratios of \( M \) to \( T \). The equation was solved numerically for a square cross-section by Steele [1954], and for other cross-sections by Imegwu [1960], who showed that the interaction diagram in the \( mt \)-plane is remarkably insensitive to cross-sectional shape.

### 6.2.4. Bending and Shear

In the problems involving combinations of bending moments, axial force and torque that have been considered so far in this section, these stress resultants are independent of one another as far as the equilibrium of the beam is concerned, and therefore their interaction may be rigorously studied on a purely local basis. In particular, for the purpose of lower-bound analysis, if a statically admissible stress distribution is found for each such resultant separately, then a linear combination of such stress distributions is also statically admissible.

If a beam is subject to transverse loading, then this loading determines both the shear force \( V \) and the bending moment \( M \), which are therefore related to each other by the equilibrium equation

\[
\frac{dM}{dz} = V.
\]

Strictly speaking, then, \( M \) and \( V \) are not generalized stresses that can be specified independently of each other.

Indeed, each case of a transversely loaded beam presents a distinct problem of limit analysis. As discussed in 6.1.2, even such closely related cases as the end-loaded cantilever and the center-loaded simply supported beam are different. If the span-to-depth ratio is sufficiently high, however, the limit
loads tend to those predicted by the elementary plastic-hinge mechanism —
that is, the overall collapse of the beam is almost completely determined by
local behavior at a critical section. It thus becomes reasonable to formulate
an approximate local yield locus, in terms of \( M \) and \( V \), that governs local
behavior, analogous to the local yield loci found above. The problem of limit
analysis of an arbitrarily loaded beam then reduces to the determination of
the critical section or sections; this problem is studied in 6.3.2.

The particular problem of the end-loaded cantilever may be used as the
test problem for studying the local yield locus, as discussed by Drucker
[1956]. Here, the problem has already been investigated from two points of
view: in 4.5.2 as the limit of elastic—plastic bending, and in 6.1.2 as a problem
in plane limit analysis. The latter approach, as was indicated there, gives
both upper and lower bounds for the collapse load of the beam. The former
approach, as is shown next, furnishes a lower bound.

**Lower Bound as Limit of Elastic-Plastic Solution**

We saw in 4.5.2 that as the elastic core shrinks under the influence of
an increasing moment, the maximum shear stress there grows until it also
reaches the yield value, after which a secondary plastic zone forms in which
\(|\tau| = \tau_Y, \sigma = 0\). Between this central plastic zone and each of the two outer
plastic zones is an elastic zone in which, going from inside to outside, the
normal stress increases linearly in magnitude from 0 to \( \sigma_Y \), and the shear
stress decreases (parabolically in a rectangular beam or in the web of an
I-beam) from \( \tau_Y \) to zero. The limiting state is, of course, one in which the
elastic zones shrink to vanishing thickness. This limit does not represent,
strictly speaking, a statically admissible stress field, since it includes a dis-
continuity in \( \tau \). Since, however, the discontinuity occurs at two isolated
points, equilibrium of any finite element is not violated and the stress distri-
bution is acceptable. If the width of the plastic zone is \( c \), then \( V = \tau_Y bc \) and
\( M = \sigma_Y b (h^2 - c^2)/4 \). Defining \( v = V/V_U \), where \( V_U = \tau_Y A \), we immediately
obtain the dimensionless interaction curve given by

\[
m = 1 - v^2.
\]

This curve has clearly the same form as the one previously found for the
interaction between moment and axial force, with \( v \) replacing \( p \). In fact, it
can be shown that for any doubly symmetric section the yield locus in the
\( m v \)-plane is given parametrically by

\[
m = \tilde{m}(\eta), \quad v = \tilde{p}(\eta),
\]

where \( \tilde{m} \) and \( \tilde{p} \) are the same functions as in Equation (6.2.6). The proof is
based on the observation that the stress distribution for combined bending
moment and axial force, shown in Figure 6.2.5(a), is the superposition of
those shown in Figure 6.2.5(b) and (c). However, the stress distribution
(b) alone produces the same moment as (a), and the stress distribution (c) alone produces the same axial force as (a). If, now, (c) represents a block of shear stress rather than normal stress, then it produces a shear force, rather than an axial force, by exactly the same formula. When suitably nondimensionalized, therefore, the shear force is exactly the same function of the depth of the central zone as is the axial force.

\[
\begin{align*}
(a) & = (b) + (c) \\
\text{(a)} & \quad \text{(b)} \quad \text{(c)}
\end{align*}
\]

**Figure 6.2.5.** Fully plastic stress distribution at a beam cross-section under combined bending moment and axial force.

In a rectangular cantilever of length \( L \) carrying a concentrated transverse force \( F \) at its tip, collapse occurs when a plastic hinge forms at the built-in end. At this point \( M = FL \) and \( V = F \). Defining \( f = FL/M_U \), we find that the lower bound based on Figure 6.2.5(b-c) predicts

\[
f = 1 - \left( \frac{\sigma_Y}{2\tau_Y} \frac{h}{2L} \right)^2 f^2,
\]
a quadratic equation for \( f \) that can be solved explicitly for \( f \). Assuming the Tresca criterion \( (\sigma_Y = 2\tau_Y) \) and defining \( \delta = h/2L \), we can write the solution as

\[
f = \frac{1}{2\delta^2} \left( \sqrt{1 + 4\delta^2} - 1 \right). \tag{6.2.12}
\]

**Upper Bound: Hodge Approach**

The generalized strain rates conjugate to \( M \) and \( V \) are the curvature rate \( \dot{\kappa} \) and the shear rate \( \dot{\gamma} \), respectively. The strain-rate components conjugate to the stresses \( \sigma \) and \( \tau \) are \( \dot{\varepsilon} = -\dot{\kappa}y \) and \( \dot{\gamma} \). Given \( \dot{\kappa} \) and \( \dot{\gamma} \), the associated flow rule (6.2.8) produces the stresses

\[
\sigma = -\frac{\sigma_Y}{\lambda} \dot{k}y, \quad \tau = \frac{\tau_Y}{\lambda} \dot{\gamma}.
\]

Satisfaction of the yield criterion (3.3.5) requires

\[
\dot{\lambda} = \sqrt{(\tau_Y \dot{\gamma})^2 + (\sigma_y \dot{k}y)^2} = \tau_Y \dot{\gamma} \sqrt{1 + \nu y^2},
\]
where \( \nu = \sigma_Y \dot{\kappa}/\tau_Y \). The stresses are therefore
\[
\sigma = -\frac{\nu \sigma_Y y}{\sqrt{1 + \nu^2 y^2}}, \quad \tau = \frac{\tau_Y}{\sqrt{1 + \nu^2 y^2}}.
\]
(6.2.13)
The stress resultants are accordingly given in terms of the parameter \( \nu \) as
\[
M = \nu \sigma_Y \int_A \frac{y^2}{\sqrt{1 + \nu^2 y^2}} \, dA,
\]
\[
V = \tau_Y \int_A \frac{1}{\sqrt{1 + \nu^2 y^2}} \, dA.
\]

Figure 6.2.6. I-beam: geometry.

Hodge [1957b] evaluated the integrals in the preceding equations for the I-beam shown in Figure 6.2.6. If \( c/b \) and \( a/h \) are small, then the parametric form of the yield locus in dimensionless form is, after neglecting terms of order higher than the first in these quantities,
\[
m = \frac{2 \tanh \omega + j (\coth \omega - \omega \text{csch}^2 \omega)}{2 + j},
\]
\[
v = \frac{\text{sech} \omega + j \omega \text{csch} \omega}{1 + j},
\]
where \( \omega = \sinh^{-1}(\nu h/2) \), while \( j = ch/2ab \) is a dimensionless shape parameter. The limiting cases \( j = 0 \) and \( j = \infty \) correspond, respectively, to the ideal I-beam (with finite flange thickness but negligible web thickness) and the rectangular beam. The calculated interaction curves are shown in Figure 6.2.7.

Hodge also derived the same stress distribution by solving the constrained extremum problem, as in 6.2.3. On this basis Hodge regards the yield locus as also providing a lower bound. However, the shear stress given by \((6.2.13)_2\)
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Rectangular beam \((j = \infty)\)

Ideal I-beam \((j = 0)\)

I-beam \((j = 1)\)

Figure 6.2.7. Interaction curves for I-beam (Hodge [1959]).

does not vanish at the extreme fibers, and it therefore appears difficult to accept the stress distribution as statically admissible.

Plots of the lower-bound curve defined by Equation (6.2.12), the upper-bound curve due to Hodge [1957b], and the lower-bound and upper-bound curves due to Drucker [1956] (see 6.1.2) are shown in Figure 6.2.8. It can be seen that all the curves have the limit \(f = 1\) as \(\delta \rightarrow 0\), so that the effect of shear is negligible for beams that are not overly deep, and that all tend asymptotically to the hyperbola \(f = 1/\delta\) for \(\delta\) large. It must be remembered, however, that values of \(\delta\) greater than about 0.5 cannot reasonably be regarded as describing beams.

Exercises: Section 6.2

1. Using the assumptions of elementary beam theory, derive an expression for the internal virtual work in a beam of arbitrary cross-section subject to a variable bending moment \(M\) and axial force \(P\).

2. Find the ultimate yield locus for a beam having the idealized section of Figure 3.5.2.

3. For a beam whose cross-section is an isosceles triangle, find the ultimate yield loci for combined axial force and bending moment (a) perpendicular and (b) parallel to the axis of symmetry.

4. Analyze the collapse of a semicircular arch of ideal sandwich section
Figure 6.2.8. Lower-bound and upper-bound curves under combined bending and shear of a rectangular beam, load against depth-span ratio.

that is built in at the supports and carries a concentrated load $2F$ at the vertex.

5. Analyze the collapse of a symmetric simply supported arch of ideal sandwich section, forming a circular segment of angle $2\alpha$, and loaded by (a) a concentrated load $2F$ at the vertex, (b) a uniform vertical load of intensity $q$, and (c) a uniform radial load of intensity $q$.

6. Show that for combined torsion and axial force or bending moment of a circular bar, the unit circles are lower bounds to the yield loci in both the $pt$- and $mt$-planes.

7. Find a result analogous to (6.2.12) for an end-loaded cantilever of circular cross-section with radius $a$, letting $\delta = a/L$.

8. Evaluate the integrals following Equation (6.2.13) for a rectangular cross-section. Compare with the cited result of Hodge [1957b] for $j = \infty$.

Section 6.3 Limit Analysis of Trusses, Beams and Frames

6.3.1. Trusses

A truss is an assemblage of stiff bars that are more or less flexibly connected to one another at their ends. In an ideal truss, the connection is through frictionless pins, with the center of each pin coinciding with the intersection
of the centroidal lines of the bars meeting there. Moreover, all the loads are applied at the joints. Consequently, all bars carry axial forces only, and if \( n \) is the number of bars, then the bar forces \( P_1, \ldots, P_n \) in effect constitute the stress field.

In order for each pin to be in equilibrium, the vector sum of all the bar forces in the members connected through that pin must be zero. If the number of joints is \( j \), then the total number of joint equilibrium equations is \( 3j \), unless all the bars and loads lie in the same plane, in which case the truss is a plane (or planar) truss and the number of equations is \( 2j \). A nonplane truss is usually called a space truss.

The unknowns of the problem, in addition to the bar forces, are the reaction components, \( s \) in number. If \( n + s = k j \) (where \( k = 2 \) or \( 3 \) in the plane truss or space truss, respectively), then the equilibrium equations are precisely enough to determine the unknowns, and the truss is \textit{statically determinate} (or \textit{isostatic}). If \( r = n + s - kj \) is positive, then the truss is \textit{statically indeterminate} (or \textit{hyperstatic} or \textit{redundant}), and \( r \) is called the \textit{degree of static indeterminacy} (or, more simply, the \textit{indeterminacy number} or \textit{redundancy number}). If this number is negative, then the truss is unstable (or \textit{hypostatic}) and is, in fact, a mechanism. In a stable truss, the number \( s \) of reaction components must be at least equal to the number of equilibrium equations for the truss as a whole, three for the plane truss and six for the space truss. If \( s \) is greater than this number, then the truss is \textit{externally indeterminate}.

Some simple trusses, with \( r \) equal to 0, \(-1\) and \( 1 \) are shown in Figure 6.3.1. In particular, the truss in (e) is statically indeterminate of degree one even though, apparently, \( r = 6 + 4 - 2 \cdot 4 = 2 \). In fact, bar \( AD \) cannot deform.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{simple_trusses.png}
\caption{Some simple trusses.}
\end{figure}
and therefore cannot carry any force. This bar may, as a result, be ignored in any analysis of the truss.

**Limit Analysis of Trusses**

A truss member will be said to fail if it can undergo significant lengthening or shortening with no significant change in the bar force. Failure in this sense can result from yielding, if the material is perfectly plastic or nearly so, or, in the case of a compression member, from buckling. Since the bar force in a failed member is no longer determined by equilibrium but by the failure criterion, it can be presumed as known if the properties of the bar are known, and the number \( n \) of unknown bar forces drops by one, as does the indeterminacy number \( r \). The truss therefore becomes unstable if \( r + 1 \) bars fail. In particular, a statically determinate truss collapses as soon as one bar fails.

Any choice of \( r + 1 \) bars that fail provides a mechanism with one degree of freedom that can be used with the upper-bound theorem. Given a reference velocity \( v \), the elongation rates \( \Delta_i \) of the \( i \)th failed bar may be determined by geometry. Let \( P^+_i \) denote the ultimate bar force in tension, and \( P^-_i \) the magnitude of that in compression; the latter is the lesser of the yield force and the buckling force. The internal dissipation in a failed bar is \( P^+_i \Delta_i \), the superscript sign being that of \( \Delta_i \).

The truss of Figure 4.1.1 has three bars, six reaction components, and four joints, so that \( r = 1 \) and two bars must fail for collapse to occur. In 4.1.4 we read, however, that all three bars must fail for this truss to collapse. With \( P_U \) the same in all three bars, the collapse load in this case is

\[
F_U = P_U^+(1 + 2 \cos \alpha).
\]

If we were to assume a mechanism in which bars \( AD \) and \( CD \) fail, while \( BD \) remains rigid, the external work rate would be zero and the upper-bound theorem would fail to give a finite upper bound to \( F_U \). Consider, now, an asymmetric mechanism with bars \( BD \) and \( CD \) only failing. Bar \( AD \) then rotates rigidly about its support, and if the downward component of the velocity of pin \( D \) is \( v \), then its leftward component is \( v \cot \alpha \). The velocity component in the direction of bar \( CD \), equal to its elongation rate, is

\[
v(\cos \alpha + \cot \alpha \sin \alpha) = 2v \cos \alpha.
\]

Equating the external work rate to the total internal dissipation,

\[
Fv = P_U^+(1 + 2 \cos \alpha),
\]

leads to the an upper bound equal to \( F_U \). The equality can be explained as follows: suppose that the symmetry of the system is disturbed ever so slightly (e.g., by giving the load a small leftward component or by making bar \( CD \) just a little bit weaker than \( AD \)). The correct collapse mechanism would then, indeed, be the asymmetric one just considered. But a minimal asymmetry should not significantly affect the collapse load calculated on
the basis of assumed symmetry, and hence the collapse loads given by the symmetric and asymmetric mechanisms should be the same.

Let us now look at the possible collapse mechanisms of the truss of Figure 6.3.1(d) under a downward force $F$ applied at joint $C$. First, it is apparent that any mechanism in which bars $AC$ and $CD$ do not deform does not permit any translation of $C$, so that the force $F$ cannot do any work; such a mechanism is not admissible. A mechanism in which both $AC$ and $CD$ fail but the other bars remain rigid allows rotation of $BC$ about $B$, leading to a zero external work rate for the given loading. The only mechanisms to be considered, therefore, are those in which either $AC$ or $CD$ fails, but not both; these mechanisms number six, and are shown in Figure 6.3.2. Dashed lines represent bars that fail.

We assume that the bars are of identical cross-section and sufficiently stiff so that $P_U^+ = P_U^- = P_U$ in every bar. The mechanism (i) in which $CD$ and $BC$ fail presents a rigid rotation of $AC$ about $A$; if the downward velocity of $C$ is $v$, then the angular velocity of rotation of $AC$ is $v/a$, where $a$ is the length of the nondiagonal bars, so that the shortening rate of $CD$ and the lengthening rate of $BC$ are also $v$. The total internal dissipation is thus $2P_Uv$, and the resulting upper bound on $F$ is $2P_U$. If (ii) $CD$ and $AB$ fail, then $AC$ rotates about $A$, $BD$ rotates about $D$, and $BC$ rotates about its midpoint. The shortening rate of $CD$ is again $v$, as is the lengthening rate of $AB$, and the upper bound is once more $2P_U$.

The mechanisms in which $AC$ fails along with either (iii) $BC$ or (iv) $AB$, and the one (v) in which $CD$ and $BD$ fail, all give the same upper bound of $(1 + 1/\sqrt{2})P_U$. Finally, in the shear mechanism (vi) in which $AC$ and $BD$ fail, these bars lengthen and shorten, respectively, at the rate $v/\sqrt{2}$.

Figure 6.3.2. Collapse mechanisms for the truss of Figure 6.3.1(d).
leading to the upper bound $\sqrt{2}P_U$. This upper bound, being the smallest, must equal the ultimate load.

To confirm this equality, we analyze the truss statically under this value of $F$, assuming that $P_{AC} = P_U$ and $P_{BD} = -P_U$. We find that all the other bar forces are of magnitude $P/\sqrt{2}$, so that the “stress field” (the distribution of bar forces) is plastically admissible.

It is clear that if the cross-sectional areas of all three nondiagonal bars were to be reduced by a factor of $\sqrt{2}$, their ultimate bar forces would fall to $1/\sqrt{2}$ times those of the diagonals, and the stress field would still be — just barely — plastically admissible under the just-calculated collapse load. Collapse would then occur with all bars yielding, indicating the most efficient use of material; such a truss would be an example of minimum-weight design.

A kinematic analysis of the minimum-weight truss shows that all six mechanism based on failure in two bars produce the same upper bound. The most general collapse mechanism, then, is a linear combination of the six. It is easy to see that this mechanism has four degrees of freedom, represented by the horizontal and vertical displacements of joints $B$ and $C$.

Limit Design of Trusses

The design of structures on the basis of limit analysis is known as limit design. Limit design for minimum weight is particularly simple for trusses in which no bars buckle, because the weight of a bar is directly proportional to its area, and hence to its strength. Let the ultimate bar force of the $i$th bar be $P_{Ui}$, and its length $L_i$. In minimum-weight design, the object is then to minimize

$$\sum_{i=1}^{n} P_{Ui}L_i$$

subject to inequality constraints on the $P_{Ui}$, ensuring that the truss is strong enough to carry the prescribed load. The problem is one of linear programming; it is discussed in its general form after a study of the present example.

Suppose, for simplicity, that $P_{U(AB)} = P_{U(BC)} = P_{U(CD)} = P_U$ and $P_{U(AC)} = P_{U(BD)} = P_{U2}$. We are thus led to minimize

$$3P_U + 2\sqrt{2}P_{U2}$$

subject to the following inequalities:

$$2P_U \geq F, \quad P_U + \frac{P_{U2}}{\sqrt{2}} \geq F, \quad \sqrt{2}P_{U2} \geq F;$$

in this elementary case the six mechanisms furnish only three independent inequalities, and the second of the three is redundant, since it is satisfied.
whenever the other two are. The solution is simple, as we already know: all the inequalities are satisfied as equalities when \( P_{U2} = \sqrt{2}P_{U1} = F/2 \).

**Linear-Programming Formulation of Limit Design**

If a truss has \( n \) bars, then the number of different combinations of \( r + 1 \) bars — that is, the number of different mechanisms — is

\[
m = \frac{n!}{(n-r-1)!(r+1)!}.
\]

Let \( v \) denote a positive reference velocity in the \( k \)th mechanism \((k = 1, \ldots, m)\), such that the elongation rate of the \( i \)th bar in this mechanism is \( \alpha_{ik}v \), and the velocity conjugate to the applied force \( F_j \) \((j = 1, \ldots, l)\) is \( \beta_{jk}v \). According to the upper-bound theorem, then,

\[
\sum_{i=1}^{n} \alpha_{ik}P_{Ui} \geq \sum_{j=1}^{l} \beta_{jk}F_{j}, \quad k = 1, \ldots, m. \tag{6.3.1}
\]

The problem of minimum-weight design of a truss is then one of minimizing the linear function

\[
G(P_{U1}, \ldots, P_{Un}) = \sum_{i=1}^{n} L_{i}P_{Ui}
\]

subject to the linear constraints (6.3.1); this problem is one of linear programming. The connection between limit design and linear programming was apparently first noted by Heyman [1951].

The standard linear-programming problem is one of maximizing, rather than minimizing, the *objective function* \( c^T x \), where \( c \) and \( x \) are \( 1 \times n \) column matrices, subject to the constraints

\[
a_{k}^T x \leq b_{k}, \quad k = 1, \ldots, m, \tag{6.3.2}
\]

where the \( a_k \) are \( 1 \times n \) column matrices, and the \( b_k \) are real numbers. Often the additional constraint that the \( x_i \) be nonnegative is imposed. The problem of minimum-weight design of a truss becomes a standard problem if we identify \( x_i \) with \( P_{Ui} \), \( c_i \) with \( -L_i \), \( a_k \) with the \( k \)th column of the matrix \(-[\alpha_{ik}]\), and \(-b_k \) with the right-hand side of (6.3.1). By defining the \( m \times n \) matrix \( A \) through \( A^T = [a_1, \ldots, a_m] \) and the \( m \times 1 \) matrix \( b \) through \( b = (b_1, \ldots, b_m) \), inequality (6.3.2) may be rewritten in the short form

\[
Ax \leq b, \tag{6.3.3}
\]

and the additional constraint as

\[
x \geq 0. \tag{6.3.4}
\]
In an alternative formulation of the linear-programming problem, the constraint conditions (6.3.3)–(6.3.4) are replaced by

$$Ax = b, \quad x \geq 0. \quad (6.3.5)$$

An efficient numerical solution method due to Dantzig, called the simplex algorithm, exists for the problem in this form (see Dantzig [1963]).

The constraints (6.3.3)–(6.3.4) can be converted to the form (6.3.5) by introducing the slack variables $y_k$ ($k = 1, \ldots, m$), which form the $m \times 1$ matrix $y$, and rewriting (6.3.3) as

$$Ax + y = b, \quad y \geq 0. \quad (6.3.6)$$

The $(n + m) \times 1$ matrix $\bar{x}$ is now defined by $\bar{x}_i = x_i$ ($i = 1, \ldots, n$) and $\bar{x}_{n+k} = y_k$ ($k = 1, \ldots, m$). The $m \times (n + m)$ matrix $\bar{A}$ is similarly defined by $\bar{A}_{ki} = A_{ki}$ ($i = 1, \ldots, n$) and $\bar{A}_{k,n+j} = \delta_{jk}$ ($j = 1, \ldots, m$). The constraint inequalities (6.3.4) and (6.3.6) together can now be written as

$$\bar{A} \bar{x} = b, \quad \bar{x} \geq 0,$$

a form identical with (6.3.5).

### 6.3.2. Beams

Any transversely loaded beam, except an ideal sandwich beam, is statically indeterminate in the sense of Section 4.1 — that is, the stress field cannot be deduced from the loading independently of unknown properties: at any section, an infinity of stress distributions can be found that give the same resultant moment $M$ and shear force $V$. It is conventional, however, to call a beam statically determinate (or indeterminate) if it is externally determinate (or indeterminate) in the same sense as a truss, that is, if the number of independent reaction components is the same as (or greater than) the number of equilibrium equations available to determine them. In the absence of internal hinges, this number is three for plane bending. Any hinge, whether frictionless or a plastic hinge, provides an additional equilibrium equation: at a frictionless hinge, $M = 0$, since such a hinge cannot transmit moment, while at a plastic hinge $M = M_U^+$ or $M = -M_U^-$. The indeterminacy number of a beam is accordingly $r = s - h - 3$, where $s$ is the number of reaction components and $h$ is the number of hinges. Like a plane truss, the beam collapses when $r$ is reduced to $-1$, so that if $h_0$ hinges are present initially, the number of plastic hinges required for collapse is $s - h_0 - 2$, and specifically, one if the beam is statically determinate, and two or more if it is statically indeterminate. A plastic hinge may form at any point of the beam at which the condition $|M| = M_U$ is possible, that is, in the interior of a

---

1Provided that these components do not include three collinear forces.
span, at a built-in end, or at an intermediate support. A collapse mechanism is admissible if it does not violate any support condition (possibly relaxed by the formation of a plastic hinge) and if it produces a positive external work rate.

If the effect of shear on the formation of a plastic hinge can be neglected, as will be assumed, then a hinge that has rotated by an angle $\Delta \theta$ can be thought of as the limit of a small segment, of length, say, $\Delta x$, in which the curvature is $\Delta \theta / \Delta x$ and the plastic dissipation per unit length is $M_U |\Delta \dot{\theta}| / \Delta x$. The total internal dissipation in the hinge is therefore $M_U |\Delta \dot{\theta}|$.

A moment distribution is statically and plastically admissible if it is in equilibrium with the applied loads, is consistent with all force and moment end conditions and frictionless hinge conditions (if any), and is such that $|M| \leq M_U$ everywhere. In the moment distribution at collapse, the points where $M = \pm M_U$ are precisely the ones where plastic hinges form.

**Example: Beam with Point Loads**

If the beam carries point loads only, then the moment can vary only in straight-line fashion between points where concentrated forces (loads or reactions) act, and therefore the actual collapse mechanism must be one in which hinges form at load points, built-in ends, or intermediate supports. Consider the beam shown in Figure 6.3.3(a), indeterminate to the first degree, and requiring two plastic hinges for collapse. The possible mechanisms are (1) with hinges at $A$ and $B$, (2) with hinges at $A$ and $C$, and (3) with hinges at $B$ and $C$. They are shown as Figure 6.3.3(b), (c), and (d), respectively.

Let us look first at mechanism 1. If the downward displacement of point $B$ is $\Delta$, then that of point $C$ is $\frac{1}{2} \Delta$, so that the external work rate is $\alpha F \Delta + \frac{1}{2} (1 - \alpha) F \Delta = \frac{1}{2} (1 + \alpha) F \Delta$. The angles of rotation of the hinges at $A$ and $B$, assumed small, are respectively $3 \Delta / L$ and $9 \Delta / 2L$; the total internal dissipation is therefore $15 M_U \Delta / 2L$, and if this is equated to the external work rate, the resulting upper bound on $FL / M_U$ is $15 / (1 + \alpha)$.

A similar analysis of mechanisms 2 and 3 leads to the respective upper bounds of $12 / (2 - \alpha)$ and $6 / (1 - \alpha)$; but the second of these is greater than the first for any $\alpha$ between 0 and 1, so that mechanism 3 may be discarded. It can easily be seen that mechanism 1 gives the lesser upper bound when $\alpha > 2 / 3$, and mechanism 2 when $\alpha < 2 / 3$. The ultimate load is therefore given by

$$\frac{F_U L}{M_U} = \begin{cases} 
\frac{12}{2 - \alpha}, & \alpha \leq \frac{2}{3}, \\
\frac{15}{1 + \alpha}, & \alpha \leq \frac{2}{3}.
\end{cases}$$

The preceding result may also be cast in the form of two inequalities,
parametrically dependent on $\alpha$, that the total load $F$ must obey:

\[
\frac{(2 - \alpha)FL}{MU} \leq 12, \quad \frac{(1 + \alpha)FL}{MU} \leq 15.
\]

Yet another way of presenting the result would be to regard the loads at $B$ and $C$ as two independent loads $F_1$ and $F_2$. The inequalities are accordingly rewritten as

\[
F_1 + 2F_2 \leq 12 \frac{MU}{L}, \\
2F_1 + F_2 \leq 15 \frac{MU}{L},
\]

represented graphically by the interaction diagram shown in Figure 6.3.3(e).

The last pair of inequalities, Equations (6.3.7), can also be derived by means of an equilibrium analysis. Suppose that a plastic hinge has already formed at $A$, with $MA = -MU$; the beam is then statically determinate, and the bending moments at $B$ and $C$ can easily be calculated to be, respectively, $(2F_1 + F_2)L/9 - 2MU/3$ and $(F_1 + 2F_2)L/9 - MU/3$. The requirement that these moments not exceed $MU$ gives precisely the inequalities (6.3.7).
The inequalities are also design criteria: they give the minimum value of $M_U$ that the section must have in order to carry a given set of loads $F_1$, $F_2$.

**Example: Beam with Distributed Load**

Suppose, now, that the beam just examined carries a uniformly distributed load of intensity $F/L$ rather than the point loads. It can still be assumed with some certainty that one of the plastic hinges necessary for collapse will form at the built-in end, but the other hinge can be, in principle, anywhere along the span of the beam. In fact it will form, of course, at the section where the bending moment has a local extremum.

If a hinge has formed at the built-in end ($x = 0$), then the bending moment at any $x$, $0 \leq x \leq L$, is

$$M(x) = M_U \left[ - \left(1 - \frac{x}{L}\right) + \frac{f}{2} \frac{x}{L} \left(1 - \frac{x}{L}\right) \right],$$

where $f = FL/M_U$. The maximum of the quantity in brackets occurs at $x/L = \frac{1}{2} + 1/f$, and equals $(f - 4 + 4/f)/8 \equiv \phi(f)$. Any value of $f$ for which $\phi(f)$ does not exceed unity is a lower bound for $f_U = F_U L/M_U$. For example, $\phi(10) = 0.8$, so that 10 is a lower bound. On the other hand, $\phi(12) = 1.04$, so that $f_U$ must be somewhat less than 12.

The actual value of $f_U$ is obtained by setting $\phi(f)$ equal to unity, giving the quadratic equation

$$f^2 - 12f + 4 = 0.$$ 

This equation has the two roots $6 \pm 4\sqrt{2}$. Clearly, since what is sought is the greatest lower bound, the larger root must be chosen. We thus obtain $f_U = 11.657$

In a kinematic solution, a mechanism with a plastic hinge at $x = 0$ and another at $x = \alpha L$, with $\alpha$ to be determined, is assumed, as in Figure 6.3.4. If the downward displacement of the hinge is $\Delta$, then the average displacement of both rigid portions of the beam is $\Delta/2$, so that the external work rate is $F\Delta/2$. The hinge at the built-in end rotates by an angle $\Delta/\alpha L$, 

![Figure 6.3.4. Uniformly loaded single-span beam, simply supported at one end and built in at the other: collapse mechanisms.](image-url)
and the other hinge by the angle $\Delta/\alpha L + \Delta/(1 - \alpha)L$. The total internal dissipation is therefore

$$\frac{2 - \alpha}{\alpha(1 - \alpha)} M_U \frac{\dot{\Delta}}{L}.$$

It follows that an upper bound to $f_U$ is

$$f = \frac{2(2 - \alpha)}{\alpha(1 - \alpha)}.$$

The least upper bound is found by minimizing $f$ with respect to $\alpha$:

$$\frac{1}{2} \frac{df}{d\alpha} = \frac{2 - \alpha}{\alpha(1 - \alpha)^2} - \frac{2 - \alpha}{\alpha^2(1 - \alpha)} - \frac{1}{\alpha(1 - \alpha)}$$

$$= \frac{\alpha(2 - \alpha) - (1 - \alpha)(2 - \alpha) - \alpha(1 - \alpha)}{[\alpha(1 - \alpha)]^2} = 0,$$

leading to the quadratic equation

$$\alpha^2 - 4\alpha + 2 = 0.$$

Since $\alpha$ must be less than 1, the only relevant root is $\alpha = 2 - \sqrt{2}$ and gives the least upper bound $f = 6 + 4\sqrt{2} = 11.657$, which, of course, coincides with the previously found greatest lower bound.

Without the analytical solution, assumed values of $\alpha$ give upper bounds that may be satisfactory; for example, $\alpha = 0.5$ leads to $f = 12$, and $\alpha = 0.6$ leads to $f = 11.667$. Moreover, assumed mechanisms can also be used to give lower bounds without resorting to an analytical solution. Consider, for example, the mechanism with $\alpha = 0.5$, corresponding to $f = 12$. The moment distribution corresponding to this mechanism is given by

$$M(x) = -M_U \left[1 - 7 \frac{x}{L} + 6 \left(\frac{x}{L}\right)^2\right],$$

and is not plastically admissible because $|M|_{\text{max}} = M(7L/12) = (25/24)M_U$. If, however, all the moments are multiplied by 24/25, then the distribution becomes plastically admissible, and in equilibrium with a load for which $f = (24/25)12 = 11.52$, which is thus a lower bound.

**Example: Continuous Beam**

A beam with two or more spans, separated by intermediate supports that exert transverse force reactions, is called a *continuous beam*. The rule governing the degree of static indeterminacy is the same for continuous as for simple beams. However, collapse of a continuous beam may occur in one span only, and does not in general require $r + 1$ plastic hinges. The collapse of a span between two intermediate supports, or between an intermediate support and a built-in end support, requires three hinges. A span between
an intermediate support and a simple end support will collapse with only two hinges. It is thus possible for a continuous beam to remain statically indeterminate at collapse. For this reason the kinematic method is preferable by far for the limit analysis of continuous beams.

![Figure 6.3.5](image.png)

**Figure 6.3.5.** Continuous beam, simply supported at one end, built in at the other, and with an intermediate support, carrying a concentrated load at the midpoint of each span.

Looking at the beam shown in Figure 6.3.5, we see immediately that the feasible collapse mechanisms are (a) the one in which span $AB$ collapses like the beam of Figure 6.3.4, and (b) the one in which span $BC$ collapses like a beam with both ends built in. In mechanism (b) hinges develop at $B$ (or rather, just to the right of $B$), at $C$, and at midspan. The loads $F_1$, $F_2$ are governed by the uncoupled inequalities,

$$F_1 \leq (6 + 2\sqrt{2}) \frac{M_U}{L}, \quad F_2 \leq 16 \frac{M_U}{L}.$$ 

These inequalities also specify the minimum value of $M_U$.

### 6.3.3. Limit Analysis of Frames

A rigid frame (or simply a frame) is an assemblage of bars that are joined together rigidly, so that they cannot rotate with respect to one another. The joints transmit bending moment, and the members resist the applied loads primarily through bending; axial force and shear are considered secondary effects. Collapse is assumed to occur when sufficient plastic hinges have formed to produce a mechanism. In a multistory frame, collapse may be limited to a single story, and therefore the overall degree of static indeterminacy is not a relevant parameter for the determination of the necessary number of hinges.

**Simple Frame**

A one-story, one-bay frame such as shown in Figure 6.3.6 is statically indeterminate of degree three, and the collapse of the frame as a whole indeed requires four hinges, as shown in Figures 6.3.6(a) and (c). Consider, however, Figure 6.3.6(d), which illustrates the beam mechanism. This mechanism does not entail collapse in the sense of unlimited displacements; the deflection of the beam is limited by that of the columns. In practice, however, a structure
may be said to collapse when its displacements can become significantly greater than those in the elastic range. If the axial elongation of the beam is neglected, a deflection $\Delta$ of the central hinge requires that the beam-column joints move laterally inward by a distance $L - \sqrt{\frac{1}{4}L^2 - \Delta^2} \approx \frac{\Delta^2}{L}$. This distance represents the elastic deflection of the columns, $\Delta_e$, which is of the same order of magnitude as the beam deflection when the whole frame is elastic. Now $\frac{\Delta}{\Delta_e} \approx \sqrt{\frac{L}{\Delta_e}}$ is a large number. We are therefore justified in regarding the beam mechanism as a collapse mechanism.

![Figure 6.3.6. One-story, one-bay frame: (a)–(c) four-hinge mechanisms; (d) beam mechanism.](image)

The only pertinent collapse mechanisms for the frame of Figure 6.3.6 are the beam mechanism (d), the panel or sidesway mechanism (c), and the composite mechanism (b), which is a superposition of (c) and (d) in which the hinge at $B$ is eliminated. The composite mechanism (a) — the mirror image of (b) — in which joint $D$ is rigid, entails negative work done by the horizontal force and therefore is viable only when this force is zero, in which case it is equivalent to (b). When a hinge is assumed to be at a joint such as $B$ or $D$, it will actually be in the weaker of the two members meeting there — that is, it forms when the bending moment (which, for equilibrium, must be the same in both members as the joint is approached) reaches the smaller of the two values of $M_U$.

Let $M_{U1}$, $M_{U2}$, and $M_{U3}$ denote the values of $M_U$ in $AB$, $BD$, and $DE$, respectively. If it is assumed that the horizontal load $F_1$ is, for example, a wind load which is just as likely to act to the left at $D$ as to the right at $B$ (so that the loading of Figure 6.3.6 represents only one of two mirror-image cases), then the frame design should be symmetric, and $M_{U3} = M_{U1}$.

The upper-bound theorem applied to the three mechanisms (b)-(d) gives the following inequalities:

$$F_2L \leq 4M_{U2} + 4\min(M_{U1}, M_{U2}),$$

$$F_1H \leq 2M_{U1} + 2\min(M_{U1}, M_{U2}),$$

$$2F_1H + F_2L \leq 4M_{U1} + 4M_{U2} + 4\min(M_{U1}, M_{U2}).$$
As before, these inequalities serve both analysis and design. For the purposes of analysis, let us assume equal values of $M_U$ for all three members. The interaction diagram between $F_1$ and $F_2$ is then as shown in Figure 6.3.7(a). The design implications of inequalities (6.3.8) are discussed in the following subsection.

![Interaction diagrams for the frame of Figure 6.3.6](image)

**Figure 6.3.7.** Interaction diagrams for the frame of Figure 6.3.6: (a) load plane; (b) design plane (see page 391).

**Complex Frames**

In a frame comprising several stories and bays, the number of possible collapse mechanisms can become quite large. Every transversely loaded member may form a beam mechanism, and each story may produce a panel mechanism. Furthermore, at any joint at which three or more members come together, a plastic hinge may form independently in each member near the joint (if only two members meet, the hinge can form only in the weaker member).

It is convenient to establish a basis of independent mechanisms, called *elementary mechanisms*, such that all mechanisms may be regarded as superpositions of the elementary ones. These elementary mechanisms, as first discussed by Neal and Symonds [1952], consist of all the beam and panel mechanisms, and in addition, of the *joint mechanisms* constituted by the formation of plastic hinges, at a joint, in every one of the members that come together there, resulting in a rotation of the joint [see Figure 6.3.8(e)]. The joint mechanisms are not in themselves collapse mechanisms, since the external work rate associated with them is zero (unless an external moment acts at the joint), but they are used in combination with beam and/or panel mechanisms in order to cancel superfluous hinges.

Let $r$ denote, as before, the degree of redundancy of the frame. A simple method of determining $r$ is to cut the frame at a sufficient number of sections so that it just becomes statically determinate, that is, equivalent to a set of simply supported beams and/or cantilevers; $r$ is then the number of stress
resultants (moments, axial forces and shear forces) that can arbitrarily be specified at the cuts. Equivalently, \( r \) is the number of sections at which the moment can be arbitrarily prescribed. Suppose, now, that the number of critical sections — that is, sections at which a plastic hinge can form — is \( n \). It follows that there are \( n - r \) independent relations among the \( n \) moments at the critical sections, and these relations are equilibrium equations. Each such equation can be associated, by means of the principle of virtual work, with a mechanism. Consequently, there are \( n - r \) independent mechanisms.

As an example, consider the two-bay frame shown in Figure 6.3.8(a). By means of two cuts, the frame can be transformed into three disconnected cantilevers, and therefore \( r = 2 \times 3 = 6 \). The critical sections, as shown, number 10. Consequently the frame has four independent mechanisms. In terms of elementary mechanisms, these are (b)–(c) the two beam mechanisms, (d) the panel mechanism, and (e) the joint mechanism.

In the method of superposition of mechanisms due to Neal and Symonds [1952], the analysis begins by determining the upper bounds predicted by the elementary beam and panel mechanisms. Because of the symmetry of the structure, the two beam mechanisms give the same upper bounds. We thus obtain the two inequalities

\[
(2F)L\dot{\theta} \leq M_0\dot{\theta} + 2M_0(2\dot{\theta}) + 2M_0\dot{\theta} = 7M_0\dot{\theta}, \quad (b, c)
\]

\[
F(2L)\dot{\theta} \leq (4\cdot M_0 + 2\cdot 1.5M_0)\dot{\theta} = 7M_0\dot{\theta}, \quad (d)
\]

both of which yield the upper bound \( F = 3.5M_0/L \).

In order to improve the upper bound, we proceed to study composite mechanisms. Mechanism (f) is a superposition of (c), (d), and (e) in which the hinges at sections 5 and 6 are eliminated, while a hinge is created at 4. The internal dissipation of the mechanisms can therefore be obtained by subtracting from the sum of the right-hand members of the two preceding inequalities the quantity \((1.5M_0 + 2M_0 - 2M_0)\dot{\theta} = 1.5M_0\dot{\theta}\), or \((7 + 7 - 1.5)M_0\dot{\theta} = 12.5M_0\dot{\theta}\). The external work rate is just the sum of the left-hand sides, or \(4FL\dot{\theta}\). Mechanism (f) therefore gives the upper bound \( F = 3.125M_0/L \).

Mechanism (g) is a further superposition of (f) and (b) in which the hinge at 2 is eliminated. The internal dissipation is \((12.5 + 7 - 1)M_0\dot{\theta} = 18.5M_0\dot{\theta}\), while the external work rate is \(6FL\dot{\theta}\). We thus obtain the even smaller upper bound of \( F = 3.083M_0/L \).

In the present example, it appears that we have run out of reasonable mechanisms, and the result should give us the collapse load. In more complex cases, it may be quite difficult to make sure that all the possible collapse

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1The circled numbers next to the members mean that the value of \(M_U\) for a member is the given number times a reference moment \(M_0\).
mechanisms have been explored. The only way to check whether the best upper bound that has been found indeed gives the collapse load is to see if it is also a lower bound, that is, to find a statically admissible moment distribution such that $|M| = M_U$ at all sections corresponding to hinges in the mechanism, and $|M| \leq M_U$ elsewhere. In the present example this is easy, since the optimal mechanism represents total collapse and thus involves $r+1 = 7$ hinges, leaving the structure statically determinate at collapse. The four independent equilibrium equations (which can be formed by applying the principle of virtual work to the elementary mechanisms) and the seven hinge conditions give eleven equations for the ten critical-section moments and the load $F$. It turns out that the moments at 2, 5, and 6 do not exceed the local values of $M_U$, and the load is in fact equal to the best upper-bound
value derived above.

The situation is more difficult in multistory frames, in which the best mechanism found by the method of superposition of mechanisms often represents partial collapse, with fewer than \( r + 1 \) plastic hinges and therefore not enough equations for a rigorous moment check. If the degree of redundancy remaining at failure is small (one or two), a trial-and-error procedure is usually applied: guesses are made for a sufficient number of critical-section moments so that the equilibrium equations can be solved. In more complicated cases, the moment-distribution method due to Horne [1954] and English [1954] may be used; for examples of application, see Hodge [1959], Chapter 3.

A method of analysis based on the lower-bound theorem is the method of inequalities, due to Neal and Symonds [1951]. The critical-section moments \( M_i \) \((i = 1, \ldots, n)\) are governed by the \( 2n \) inequalities

\[-M_{Ui} \leq M_i \leq M_{Ui},\]

and by the \( n - r \) equilibrium equations, at least one of which contains the load \( F \). The equilibrium equations can be transformed by means of linear combinations so that only one of them contains \( F \). The problem now is again one of linear programming: the equation containing \( F \) serves as the defining equation for \( F \) as the function of the \( M_i \) that is to be maximized, with the remaining equations, as well as the inequalities, serving as constraints.

### 6.3.4. Limit Design of Frames

#### Limit Design of a Simple Frame

The achievement of a minimum-weight design is not so simple for frames as it is for trusses, because there is no simple proportionality, or indeed any one-to-one relation, between weight and strength. In simple frames, such as the one analyzed in the preceding subsection, a trial-and-error approach is usually the easiest (see Heyman [1953]). In complex frames a more systematic approach, based on some simplifying assumptions, is necessary in order that the problem may be converted into one of mathematical (not necessarily linear) programming.

As a first example of limit design of a frame, we consider the fixed-base rectangular frame of Figure 6.3.6 (page 386), for which we derived inequalities (6.3.8). Let the design loads (working loads times appropriate safety factors) be such that \( F_2 L = 3F_1 H \overset{\text{def}}{=} 12\bar{M} \), where \( \bar{M} \) is a reference quantity having the dimensions of a moment. The sections chosen for the columns and the beam must then satisfy the following inequalities:

\[ M_{U2} + \min(M_{U1}, M_{U2}) \geq 3\bar{M}, \quad (a) \]
Table 6.3.1. Plastic Moduli of Selected Wide-Flange Sections

<table>
<thead>
<tr>
<th>( Z ) (in.(^3))</th>
<th>Shape</th>
<th>( Z ) (in.(^3))</th>
<th>Shape</th>
<th>( Z ) (in.(^3))</th>
<th>Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>287</td>
<td>W14×159</td>
<td>212</td>
<td>W14×120</td>
<td>150</td>
<td>W16×77</td>
</tr>
<tr>
<td>260</td>
<td>W14×145</td>
<td>198</td>
<td>W16×100</td>
<td>130</td>
<td>W16×67</td>
</tr>
<tr>
<td>234</td>
<td>W14×132</td>
<td>175</td>
<td>W16×89</td>
<td>105</td>
<td>W16×57</td>
</tr>
</tbody>
</table>

\[ M_{U1} + \min(M_{U1}, M_{U2}) \geq 2\bar{M}, \quad (b) \]
\[ M_{U1} + M_{U2} + \min(M_{U1}, M_{U2}) \geq 5\bar{M}. \quad (c) \]

The solution of these inequalities is shown in Figure 6.3.7(b) (page 387) as the “safe region” whose polygonal boundary will be called the safe boundary. Any point in the safe region or on the safe boundary will be said to represent a safe design.

We note that \( M_{U1} \) must be at least equal to \( \bar{M} \) and \( M_{U2} \) to \( 1.5\bar{M} \); but if these criteria are met, there is no need for \( M_{U1} \) to be greater than \( 2\bar{M} \), or for \( M_{U2} \) to be greater than \( 3\bar{M} \). Thus the choice of sections can be made from a rather restricted range. Let the frame dimensions be \( L = 24 \) ft. and \( H = 12 \) ft., so that the total weight of the frame is proportional to \( w_1 + w_2 \), where \( w_1 \) and \( w_2 \) denote the weight per unit length of the column and beam, respectively. The design loads will be taken as \( F_1 = 1.0 \times 10^5 \) lb and \( F_2 = 1.5 \times 10^5 \) lb, corresponding to \( \bar{M} = 3.6 \times 10^6 \) lb-in. With the usual value of \( \sigma_Y = 36 \times 10^3 \) lb/in.\(^2\) for A36 structural steel, it follows that that \( \bar{M}/\sigma_Y = 100 \) in.\(^3\). Then the ranges of the plastic modulus \( Z = M_U/\sigma_Y \) for the columns and the beam are

\[ 100 \text{ in.}^3 \leq Z_1 \leq 200 \text{ in.}^3, \quad 150 \text{ in.}^3 \leq Z_2 \leq 300 \text{ in.}^3, \]

and they must obey the inequalities

\[ 2Z_1 + Z_2 \geq 500 \text{ in.}^3, \quad Z_1 + 2Z_2 \geq 500 \text{ in.}^3. \]

It will be assumed that for architectural reasons, the section depth is to be limited to 16 in. The choice will be made from standard wide-flange sections, where the designation \( Wd \times w \) refers to a section whose depth is \( d \) (in inches) and whose weight per unit length is \( w \) (in pounds per foot). The strength of a wide-flange beam, for a given weight, increases sharply with the depth, and therefore the deepest available sections should be chosen for economy. A listing of section properties for selected wide-flange sections is shown in Table 6.3.1.

Four designs will be tried.

1. In a design with the lightest possible columns, a W16×57 section presents \( Z_1 = 105 \) in.\(^3\), requiring \( Z_{U2} \geq 290 \) in.\(^3\). Unfortunately, no standard
W16 section has a plastic modulus close to this value. We consequently choose, for the beam, a W14×159 section, giving \( Z_2 = 287 \text{ in.}^3 \). If this is assumed to be close enough, we obtain a design with \( w_1 + w_2 = 216 \text{ lb/ft} \).

2. The lightest possible beam section is a W16×77, with \( Z_2 = 150 \text{ in.}^3 \), requiring columns with \( Z_1 \geq 200 \text{ in.}^3 \); which is provided by a W16×100 \((Z = 198 \text{ in.}^3)\); this choice yields \( w_1 + w_2 = 177 \text{ lb/ft} \), a considerable improvement over the first trial.

3. The beam and column sections of design 2 can be reversed, yielding the same weight.

4. A beam section intermediate between those in designs 2 and 3 is a W16×89 and gives \( Z_2 = 175 \text{ in.}^3 \); this requires \( Z_1 \geq 162.5 \text{ in.}^3 \), and the lightest section satisfying this criterion is again W16×89; thus \( w_1 + w_2 = 178 \text{ lb/ft} \), virtually the same as designs 2 and 3, and superior to them by virtue of the greater ease of connections resulting from having the same section throughout. The minimum required value of \( Z_1 = Z_2 = 167 \text{ in.}^3 \), so that this design carries an overdesign factor of 1.05. The additional margin of safety provides an allowance for axial force. If, for example, Equation (6.2.7) is used for the interaction, then each member can carry an axial force up to \((1 - 0.85/1.05)P_U = 0.19P_U\) with no loss in moment-carrying capacity.

On the basis of design 4, \( M_{U1} = M_{U2} = 6.3 \times 10^6 \text{ lb-in} \). With the overdesign factor included, the collapse forces are \( F_1 = 1.05 \times 10^5 \text{ lb} \) and \( F_2 = 1.575 \times 10^5 \text{ lb} \) The weight of the beam is about 2000 lb, and we are therefore justified, in retrospect, in having neglected it.

Since the structure is statically determinate at collapse, an equilibrium analysis can easily be performed. The moment at the only other critical section, namely \( B \), is found to be of magnitude \( 3.78 \times 10^6 \text{ lb-in} \), so that the yield criterion is nowhere violated, and the correct mechanism was chosen. While this last result is obvious in the present example, in frames in which a large number of possible mechanisms exists, the moment check is a necessity, as in analysis.

The same equilibrium analysis shows that the beam and column \( DE \) carry compressive axial forces of \( 0.875 \times 10^5 \text{ lb} \), and that the maximum shear force (in \( CD \) and \( DE \)) has the same magnitude. We consider, first, the effect of axial force. The slenderness ratio of the columns, even if they are taken as doubly pinned, is about 20, so that they are not expected to buckle elastically. The cross-sectional area of a W16×89 section is 26.2 in.\(^2\), so that \( P_U = 9.432 \times 10^5 \text{ lb} \), and \(|P/P_U| = 0.093 < 0.19\).

We consider, finally, the effect of shear. The web area of the section is 7.9 in.\(^2\), and therefore the average shear stress in the web is some \( 11 \times 10^3 \text{ lb/in.}^2 \), well below the shear yield stress of about \( 20 \times 10^3 \text{ lb/in.}^2 \) for A36 steel. It follows that the frame can be safely designed on the basis of bending alone.
For a complex frame, the time required for a trial-and-error method of limit design would be prohibitively long. Any systematic approach is based on the assumption of a functional relation between the weight per unit length and the flexural strength (as measured by $M_U$ or $Z$) of a beam. Clearly, no such relation exists in general, but an approximate relation can be established for a limited range of sections that is used in the design of a frame. If a relation of the form

$$w = a + bM_U$$

(6.3.9)
can be found, then the problem of minimum-weight design of a frame can also be transformed into one of linear programming: the total weight is

$$W = a \sum_i L_i + b \sum_i L_i M_{Ui},$$

the summation being over all members, and therefore the objective function is

$$G(M_{U1}, \ldots) = \sum_{i=1}^{n} L_i M_{Ui},$$

independently of the parameters $a$ and $b$ in the approximate representation (6.3.9); this approach was presented by Foulkes [1953].

In the just-studied simple frame, for the three W16 sections considered, the unit weight (in lb/ft.) is very nearly given by

$$w = 5 + 0.48Z,$$

with $Z$ in cubic inches. The total weight of the frame, to within an additive constant, is therefore proportional to $Z_1 + Z_2$. The theoretical minimum-weight design is thus achieved by finding the point $(Z_1, Z_2)$ on the safe boundary where the constant-weight line $Z_1 + Z_2 = \text{const.}$ is tangent to the boundary. This point is $Z_1 = Z_2 = 167 \text{ in.}^3$, and would represent the actual minimum-weight design if a section with this value of $Z$ could be found.

Note that the point of tangency between the constant-weight line and the safe boundary is a vertex of the boundary, that is, an intersection of two of the lines forming this boundary. Since each such line represents a particular collapse mechanism, the theoretical minimum-weight frame can collapse in either of two mechanisms, or in a linear combination of the two. In particular, a linear combination of the two mechanisms, with nonnegative coefficients, can be found so that the inequality produced by the combined

\[^1\text{If the constant-weight lines are parallel to one of the boundary lines, then any point on the boundary segment of this line, including the two vertices, represents a minimum-weight design.}\]
mechanism is represented by a line that is parallel to the constant-weight lines. Such a mechanism is known as a Foulkes mechanism or a weight-compatible mechanism.

If \( n \) independent values of \( M_U \) may be used in a design, then the design space is \( n \)-dimensional. The theoretical minimum-weight design is represented by the point of tangency of a constant-weight hyperplane with the safe boundary, which is made up of intersecting hyperplanes. If the point is unique, then it is a vertex of the boundary and is therefore an intersection of at least \( n \) of the hyperplanes making up the boundary, each representing a distinct collapse mechanism. A Foulkes mechanism can be formed by combining these mechanisms linearly so that the combined mechanism is represented by a hyperplane that is parallel to the constant-weight hyperplanes.

Consider a frame design for which a Foulkes mechanism can be assumed. Let \( L_i \) denote the combined length of all the members whose ultimate moment is \( M_{U_i} \). In a Foulkes mechanism, the plastic hinges forming in these members have the property that the sum of the absolute values of their angular velocities is proportional to \( L_i \). If the proportionality factor is \( c \), then the total dissipation in the mechanism is

\[
c \sum_{i=1}^{n} M_{U_i} L_i = cG.
\]

If, moreover, a statically and plastically admissible distribution of bending moments compatible with the design can be found, then the design loads \( F_j \) are the collapse loads, and therefore, by virtual work

\[
\sum_j F_j v_j = cG,
\]

where the \( v_j \) are the velocities conjugate to the \( F_j \) in the Foulkes mechanism.

Consider, now, any other safe frame design, described by values \( M^*_{U_i} \) of the ultimate moments. Since the Foulkes mechanism is a kinematically admissible mechanism, it follows from upper-bound theorem that

\[
\sum_j F_j v_j \leq c \sum_{i=1}^{n} M^*_{U_i} L_i = cG^*,
\]

and therefore \( G^* \geq G \). This result, due to Foulkes [1954], may be stated in words as follows: A frame design that admits a Foulkes mechanism and a compatible, statically and plastically admissible distribution of bending moments is a minimum-weight design.

Upper-bound and lower-bound theorems for minimum-weight design were also established by Foulkes. The upper-bound theorem is obvious, since any
safe design provides an upper bound to the minimum weight. Conversely, any design based on a Foulkes mechanism, without necessarily satisfying the condition of admissible moments, provides a lower bound to the minimum weight. A general method, based on Foulkes’ theorems, for the minimum-weight design of highly redundant frames was first developed by Heyman and Prager [1958].

The concept of a Foulkes mechanism can be extended to structures other than frames, including those modeled as continua. If, for example, a beam can be designed with an arbitrarily varying cross-section, then a minimum-weight design, based on the relation (6.3.9), is one in which the moment distribution at collapse is such that $|M| = M_U$ everywhere. Plastic flow, in this case, is not localized in hinges but occurs throughout the beam, with a curvature-rate distribution $\dot{\kappa}$. Since the plastic dissipation in a Foulkes mechanism is proportional to $G$, it follows that

$$\int_0^L M_U |\dot{\kappa}| \, dx = c \int_0^L M_U \, dx,$$

and the Foulkes mechanism is one in which $|\dot{\kappa}| = c$.

Similarly, a Foulkes mechanism for a plate obeying the Tresca criterion is one in which

$$|\dot{\kappa}_1| + |\dot{\kappa}_2| + |\dot{\kappa}_1 + \dot{\kappa}_2| = 2c,$$

where $\dot{\kappa}_1$ and $\dot{\kappa}_2$ are the principal curvature rates.

The general criterion for minimum-weight design of continua is due to Drucker and Shield [1957]. Applications to plate design were discussed by Hopkins and Prager [1955], Prager [1955b], and Freiberger and Tekinalp [1956], and to shells by Onat and Prager [1955], Freiberger [1956b] and Onat, Schumann, and Shield [1957]. The results for variable-section beams were applied to frame design by Heyman [1959, 1960] and by Save and Prager [1963]. Further contributions are due to Chan [1969], Maier, Srinivasam, and Save [1972], and Munro [1979]. Many of these results are reviewed in the books by Neal [1963], Massonnet and Save [1965], Heyman [1971], Save and Massonnet [1972], Rozvany [1976], Horne [1979], and Borkowski [1988].

**Additional Remarks**

1. In both the analysis and the design of frames, all loads were assumed to be concentrated, thus fixing in advance the locations of the critical sections. If any span carries a distributed load, then the critical section in that span must be assumed, and therefore any mechanism gives an upper bound to the collapse load; improvements to the upper bound can be achieved by changing the hinge locations. On the other hand, if any distributed load is replaced by a statically equivalent (equipollent) set of concentrated loads, then the collapse load calculated on the basis of the concentrated loads is a lower
bound on the collapse load for the distributed load. This result, derived by Symonds and Neal [1951], is known as the load-replacement theorem.

2. Only rectangular frames were studied as examples. Frames with inclined members, such as the gable frame of Figure 6.3.9, can be studied analogously. The frame shown has three independent mechanisms; these can

![Figures 6.3.9](image-url)

*Figure 6.3.9. Gable frame: (a) geometry and loading; (b) beam mechanism; (c) panel mechanism; (d) gable mechanism; (e)–(f) other panel mechanisms; (h) composite mechanism.*

be taken as the beam mechanism (b), the panel mechanism (c), and the gable mechanism (d), which may also be regarded as a kind of panel mechanism. Other panel mechanisms, each of which is a combination of (c) and (d) with one hinge eliminated, are shown as (e)–(g), and a composite mechanism as (h).

3. A relation between weight and strength given by (6.3.9) can be a reasonable approximation to the properties of actual sections if their number is small. For a larger range of sections, a better approximation is usually obtained with a nonlinear relation such as \( w \propto M_0^\alpha \), where \( \alpha = 2/3 \) for geometrically similar sections, and \( \alpha = 0.6 \) gives a good approximation for many standard I-beam sections. The problem of minimum-weight design then becomes one of nonlinear programming. The computational implementation of nonlinear programming is not yet at a point where it can be readily
applied to the plastic design of complex structures.

**Exercises: Section 6.3**

1. The truss of Figure 6.3.1(d) is subject to equal horizontal forces $F$ acting to the right at $B$ and $C$, and a downward vertical force $2F$ acting at $B$. If the ultimate bar force $P_U$ is the same in all bars and is equal in tension and compression, find the smallest upper bound for $F_U$ by analyzing mechanisms, and verify that it is also a lower bound.

2. Find a minimum-weight design for the truss of the Exercise 1, assuming that $P_U = \pm \sigma_Y A$ in every bar.

3. A beam of span $L$, built in at the left end and simply supported at the right end, carries a load $\alpha F$ uniformly distributed over the left half and a concentrated load $(1 - \alpha)F$ at a distance $L/4$ from the right end.
   
   (a) Find $F_U$ as a function of $\alpha$ by means of both a kinematic and a static analysis.

   (b) Show how some reasonable upper and lower bounds can be found by means of assumed mechanisms.

   (c) Find the minimum value of $M_U$ as a function of $\alpha$.

4. In a continuous beam like that of Figure 6.3.5, the loads $F_1$ and $F_2$ are uniformly distributed over the respective spans rather than concentrated. Plot an interaction diagram between $F_1 L/M_U$ and $F_2 L/M_U$.

5. If the rate of deflection of a beam is denoted $v(x)$ and the curvature rate is $\dot{\kappa} = v''(x)$, show how a Foulkes mechanism for a beam built in at both ends can be generated by means of a deflection curve made up of parabolic arcs. Find a minimum-weight design for an ideal sandwich beam of uniform depth but variable flange area.

6. An asymmetric frame is shown in the adjacent figure.

   (a) Assuming $M_U$ to be the same in all members, find $F_U$.

   (b) Assuming only that the value of $M_U$ in the short column is one-half of that in the long column, find a minimum-weight design for the frame.
7. A two-story one-bay frame has the geometry and loading shown in the adjacent figure, and is assumed to collapse under the loading.

(a) Find the value of $M_U$ if this value is the same in all members.

(b) Find the value of $M_U$ if this is the value of the ultimate moment in the lower columns and beam, while in the upper columns and beam this value is 0.5$M_U$.

(c) Assuming only that $M_U$ has one value (say $M_{U1}$) in the lower columns and beam and another value (say $M_{U2}$) in the upper columns and beam, find the minimum-weight design, and plot the design diagram relating $M_{U1}/FL$ and $M_{U2}/FL$.

Section 6.4 Limit Analysis of Plates and Shells

The concept of plate yielding was already developed in Article 5.2.3, with the yield criterion given in terms of the moments $M_{\alpha\beta}$. As shown in 6.4.1, these moments serve as the generalized stresses in the limit analysis of plates.

In shells (as in arches), the equilibrium of moments is coupled with membrane forces (analogous to axial forces), and therefore both moments and membrane forces appear as generalized stresses. With the greater number of generalized stresses, the yield loci become more difficult to represent and approximations leading to piecewise linear yield criteria often become necessary. The theory is developed in 6.4.2, and examples are studied in 6.4.3. Thorough coverage of the material in this section can be found in the book by Save and Massonnet [1972].

6.4.1. Limit Analysis of Plates

Plastic Flow of Plates

With the stresses given by Equation (5.2.9), the plastic dissipation per unit volume at plastic collapse (or, equivalently, under the hypothesis of rigid-plastic behavior) is

$$D_p = \sigma_{\alpha\beta}\varepsilon_{\alpha\beta} = -\frac{4}{h^2}M_{\alpha\beta}\text{sgn} x_3(-x_3\kappa_{\alpha\beta}) = \frac{4}{h^2}|x_3|M_{\alpha\beta}\kappa_{\alpha\beta},$$

where $h$ is the plate thickness. Integrating through the thickness, we obtain the plastic dissipation per unit area,

$$\bar{D}_p = M_{\alpha\beta}\kappa_{\alpha\beta}. \quad (6.4.1)$$
Now consider any stress distribution $\sigma^*_{\alpha\beta}$ that does not violate the yield criterion and gives zero membrane forces. If the moments resulting from $\sigma^*_{\alpha\beta}$ are $M^*_{\alpha\beta}$, then the inequality (3.2.4) expressing the principle of maximum plastic dissipation can be integrated through the thickness, giving

$$(M_{\alpha\beta} - M^*_{\alpha\beta})\dot{\kappa}_{\alpha\beta} \geq 0.$$  \hspace{1cm} (6.4.2)

The moments and the curvature rates are thus respectively the generalized stresses and generalized strain rates for the analysis of the bending collapse of plates. Since yield criteria in terms of moments have already been formulated (see 5.2.2), the associated flow rule can be deduced from (6.4.2) to have the form

$$\dot{\kappa}_{\alpha\beta} = \lambda \frac{\partial f}{\partial M_{\alpha\beta}}.$$  

For the Mises, Tresca, and Johansen criteria, respectively, we obtain the plastic dissipations per unit area as

$$\bar{D}_p(\dot{\kappa}) = \frac{2}{\sqrt{3}} M_U \sqrt{\dot{\kappa}_1^2 + \dot{\kappa}_1 \dot{\kappa}_2 + \dot{\kappa}_2^2} \quad \text{(Mises)},$$

$$\bar{D}_p(\dot{\kappa}) = \frac{1}{2} M_U (|\dot{\kappa}_1| + |\dot{\kappa}_2| + |\dot{\kappa}_1 + \dot{\kappa}_2|) \quad \text{(Tresca)},$$

$$\bar{D}_p(\dot{\kappa}) = M_U (|\dot{\kappa}_1| + |\dot{\kappa}_2|) \quad \text{(Johansen)}.$$  

With moments replacing stresses and curvature rates replacing strain rates, the theorems of limit analysis can be applied to the estimation of collapse loads of plates as in any other problems. As with the plane problems studied in the preceding section, much more use can be made of the upper-bound theorem than of the lower-bound theorem. 

**Hinge Curves**

A hinge curve or yield curve (hinge line or yield line if straight) is a curve in the region $A$ occupied by the middle plane, across which the slope of the deflection $w$ is discontinuous. Let $\Delta \theta$ denote the change in slope encountered along a line normal to the curve, and suppose that this change takes place uniformly over a narrow zone of width $\delta$. Within this zone, the numerically larger principal curvature rate, say $\dot{\kappa}_1$, is given by $|\dot{\kappa}_1| = \Delta \theta / \delta$, with $|\dot{\kappa}_1| \gg |\dot{\kappa}_2|$. If we integrate $\bar{D}_p(\dot{\kappa})$ through the width of the zone, we obtain, for the plastic dissipation per unit length of the hinge curve, $(2M_U/\sqrt{3})|\Delta \theta|$ for the Mises criterion and $M_U|\Delta \theta|$ for the Tresca and Johansen criteria. The total internal dissipation needed in limit analysis,

$$D_{int} = \int_R D_p(\hat{\kappa})dV,$$
is therefore
\[
D_{int} = \frac{2}{\sqrt{3}} M_U \left[ \int_A \sqrt{\dot{\kappa}_1^2 + \dot{\kappa}_2^2} \ dA + \int_{HC} |\Delta \dot{\theta}| \ ds \right] \quad \text{(Mises)},
\]
\[
D_{int} = M_U \left[ \int_A \frac{1}{2} (|\dot{\kappa}_1| + |\dot{\kappa}_2| + |\dot{\kappa}_1 + \dot{\kappa}_2|) \ dA + \int_{HC} |\Delta \dot{\theta}| \ ds \right] \quad \text{(Tresca)},
\]
\[
D_{int} = M_U \left[ \int_A (|\dot{\kappa}_1| + |\dot{\kappa}_2|) \ dA + \int_{HC} |\Delta \dot{\theta}| \ ds \right] \quad \text{(Johansen)},
\]
where $HC$ denotes the hinge curve.

Comparing the Mises and Tresca results, we note that the expressions inside the square brackets coincide if
\[
\frac{1}{2} (|\dot{\kappa}_1| + |\dot{\kappa}_2| + |\dot{\kappa}_1 + \dot{\kappa}_2|) = \sqrt{\dot{\kappa}_1^2 + \dot{\kappa}_1 \dot{\kappa}_2 + \dot{\kappa}_2^2},
\]
and this occurs if and only if one of three conditions $\dot{\kappa}_1 = 0$, $\dot{\kappa}_2 = 0$, or $\dot{\kappa}_1 + \dot{\kappa}_2 = 0$ is met. Consequently, an upper-bound load obtained for a Tresca plate on the basis of a velocity field obeying one of these conditions almost everywhere\(^1\) serves for a Mises plate as well if multiplied by $2/\sqrt{3}$.

**Yield-line theory** was developed by Johansen [1932] for the ultimate-load design of reinforced-concrete slabs. In a polygonal plate, the yield curves are yield lines and divide the plate into portions that move as rigid bodies, so that all the dissipation takes place on the yield lines. If any edge is clamped, then either the portion of the plate adjacent to it does not move, or the edge itself becomes a yield line. In accordance with the upper-bound theorem, the yield-line pattern must be such as to minimize $D_{int}/D_{ext}$, where $D_{ext} = \int_A q \dot{w} \ dA$ is the external work rate.

**Applications of Yield-Line Theory**

As an example, consider a simply supported rectangular plate of dimensions $2a \times 2b$ under a uniform load $F/4ab$. “Simply supported” is interpreted in the traditional sense, that is, the plate deflection vanishes at the edges; such a plate is called “position-fixed” by Johnson and Mellor [1973]. According to a weaker definition of simple support, used by Johnson and Mellor, only downward deflection is prevented, and the plate is free to lift off.

For the traditional definition, the yield-line pattern may be one of the two shown in Figure 6.4.1.

Actually, pattern (a) is a special case of pattern (b), with $c = a$, but because of its relative simplicity it is instructive to study it separately. It can easily be shown that, if $v_0$ is the center velocity, then the slope-discontinuity rate on the yield lines is $\sqrt{(v_0/a)^2 + (v_0/b)^2}$, so that, for a Tresca material,
\[
D_{int} = 4M_U v_0 \sqrt{a^2 + b^2} \sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = 4M_U v_0 \left( \frac{a}{b} + \frac{b}{a} \right),
\]
\(^1\)“Almost everywhere” means everywhere except on a set of points whose total area is zero.
Figure 6.4.1. Yield-line patterns for a simply supported rectangular plate. Pattern (a) is a special case of pattern (b), with \( c = a \).

while \( \mathcal{D}_{\text{ext}} = \text{third} Fv_0 \), so that the upper bound on \( F \) is \( 12M_U(a/b + b/a) \).

For pattern (b), we have

\[
\mathcal{D}_{\text{int}} = 4M_Uv_0 \left( \frac{c}{b} + \frac{b}{c} \right) + 2(a-c)M_U \cdot 2\frac{v_0}{b} = 4M_Uv_0 \left( \frac{a}{b} + \frac{b}{c} \right)
\]

and

\[
\mathcal{D}_{\text{ext}} = \frac{F}{4ab} v_0 \left[ \frac{1}{3} \cdot 4bc + \frac{1}{2} \cdot 4b(a-c) \right] = \frac{1}{6} Fv_0 \left( 3 - \frac{c}{a} \right).
\]

The ratio \( \mathcal{D}_{\text{int}}/\mathcal{D}_{\text{ext}} \) is a minimum at \( c \) given by

\[
\frac{c}{a} = \frac{b}{a} \left[ \sqrt{3 + \left( \frac{b}{a} \right)^2} - \frac{b}{a} \right],
\]

which can be seen to be less than one for all \( b/a < 1 \). Consequently,

\[
F \leq \frac{8}{3} M_U \frac{a}{b} \left[ \sqrt{3 + \left( \frac{b}{a} \right)^2} + \frac{b}{a} \right]^2
\]

which is less than the upper bound given by pattern (a) except in the case of the square, when the two upper bounds are necessarily equal. The preceding results hold for a plate made of a Mises material if \( M_U \) is replaced by \( 2M_U/\sqrt{3} \).

If the plate is clamped, then the same yield-line pattern as above can be assumed, provided that the edges become yield lines. It can easily be verified that the total dissipation along the edges is just equal to that on the interior yield lines, so that the upper-bound load obtained by this method is twice what it is for the simply supported plate. It is shown later, however, that a better upper-bound load can be found for clamped plates in general.

A lower bound for the simply supported rectangular plate can be obtained by assuming the statically admissible moment field

\[
m_{11} = 1 - \left( \frac{x_1}{a} \right)^2, \quad m_{22} = 1 - \left( \frac{x_2}{b} \right)^2, \quad m_{12} = -\lambda \frac{x_1 x_2}{ab},
\]
where \( m_{\alpha\beta} = M_{\alpha\beta}/M_U \), and \( \lambda \) is a constant to be determined. It can readily be verified that the center and the midpoints of the edges yield according to all three criteria. The equilibrium condition

\[
m_{\alpha\beta,\alpha\beta} = -\frac{F}{4abM_U}
\]

gives

\[
F = 8M_U \left( \frac{a}{b} + \frac{b}{a} + \lambda \right).
\]

The best lower bound is therefore obtained with the largest \( \lambda \) such that the yield criterion is not violated anywhere. This occurs, for each of the three yield criteria, if the criterion is satisfied at the corners, where \( m_{11} = m_{22} = 0 \) and \( |m_{12}| = \lambda \). Thus the largest admissible value of \( \lambda \) turns out to be 1 for the Johansen, \( \frac{1}{2} \) for the Tresca, and \( 1/\sqrt{3} \) for the Mises criterion. For a square plate \((a = b)\), we obtain upper-to-lower-bound ratios of 1, 1.2, and 1.344, respectively, for the three criteria. For the square Johansen plate, we thus have the exact collapse load \( F_U = 24M_U \). Upper and lower bounds for uniformly loaded rectangular Johansen plates with various combinations of simply supported and clamped edges have been calculated by Manolakos and Mamalis [1986].

A plate in the shape of a regular polygon will, according to yield-line theory, deform plastically into a pyramid, with yield lines on all diagonals. If we think of a circle as the limit as \( n \to \infty \) of an \( n \)-sided polygon, then it becomes clear that for a circular plate to undergo plastic flow (collapse), the entire plate must become plastic — that is, the yield condition must be met everywhere, as was assumed in Section 5.3.

**Applications of circular-plate results**

The results for the clamped circular plate derived in 5.2.3 may be applied to finding an upper-bound load for a clamped plate of arbitrary shape. Consider the largest circle that can fit into the area \( A \) occupied by the plate; we can then assume a velocity field such that this circle is a hinge circle and the material inside it collapses like a clamped circular plate, while the material outside it remains rigid. If, for example, the plate is uniformly loaded and square, then the total load that would make the plate collapse in this mode is \( 4/\pi \) times the load carried by the largest inscribed circle, or \((4/\pi) \times 5.63 \times 2\pi M_U = 45.04 M_U\); this is less, and therefore a better upper bound, than the one of \( 48M_U \) given by yield-line theory.

If a clamped plate is carrying a single concentrated load \( F \), then the upper-bound collapse load of \( 2\pi M_U \) is obtained for any inscribed circle centered at the load. It was shown by Haythornthwaite and Shield [1958] that the moment field \( M_r = -M_U, \ M_\theta = 0 \) inside the circle can be extended outside it without violating equilibrium or the yield criterion, so that \( 2\pi M_U \) is in fact the collapse load.
Section 6.4 / Limit Analysis of Plates and Shells

The same hinge-circle mechanism may be used to obtain $2\pi M_U$ as the upper bound for a concentrated load carried by a simply supported plate of arbitrary shape. Since the moment field is $M_r = 0$, $M_\theta = M_U$ inside the circle, it can be continued outside it in a discontinuous but statically and plastically admissible manner as $M_r = M_\theta = 0$, so that $2\pi M_U$ is the collapse load in this case as well (see Zaid [1959]).

All the collapse loads for the axisymmetric Tresca plate may be used as bounds on the corresponding loads for the Mises plate, by virtue of the following reasoning: (1) Any moment field that is in equilibrium with the load and that obeys the Tresca criterion represented by the largest hexagon inscribed in the Mises ellipse (as in Figure 5.2.2, page 292), obviously does not violate the Mises criterion, and therefore the load is a lower bound for the Mises collapse load. (2) All the velocity fields associated with the sides of the Tresca hexagon satisfy one of the conditions $\dot{\kappa}_1 = 0$, $\dot{\kappa}_2 = 0$, $\dot{\kappa}_1 + \dot{\kappa}_2 = 0$; consequently, the Tresca collapse load multiplied by $2/\sqrt{3}$ is an upper bound for the Mises collapse load. Therefore, if $f_M$ and $f_T$ are the ultimate values of $f = F/2\pi M_U$ for the Mises and Tresca plates, respectively, then

$$f_T \leq f_M \leq \frac{2}{\sqrt{3}} f_T.$$

In order to obtain $f_M$ exactly, we must solve the quadratic equation

$$M_r^2 - M_r M_\theta + M_\theta^2 - M_U^2 = 0$$

for $M_\theta$, making sure that the correct root is chosen, and substitute it in the equilibrium equation. Let $\rho = r/a$, and let $\phi(\rho)$ be a function such that the distributed load (assumed to be acting downward) is given by $q(\rho) = -(f M_U/a^2)\phi'(\rho)/\rho$; note that $\phi(0) = 0$ and $\phi(1) = 1$. The differential equation for $m = M_r/M_U$,

$$\rho \frac{dm}{d\rho} + \frac{1}{2} m - \sqrt{1 - \frac{3}{4} m^2} = -f \phi(\rho),$$

must be solved subject to the initial condition $m(0) = 1$, and $f_M$ is the value of $f$ for which the solution satisfies $m(1) = 0$ for a simply supported plate and $m(1) = -1$ for a clamped plate.

If the plate is uniformly loaded, then $\phi(\rho) = \rho^2$. A numerical solution of the differential equation leads to $f_M = 3.26$ for the simply supported plate and $f_M = 5.92$ for the clamped plate. These values may be compared with the respective lower bounds of 3 and 5.63, and the upper bounds of 3.46 ($= 2\sqrt{3}$) and 6.50. Other results relating to plate collapse may be found in the books by Hodge [1959], Chapter 10; Hodge [1963]; Johnson and Mellor [1973], Chapter 15; and Save and Massonnet [1972].
6.4.2. Limit Analysis of Shells: Theory

A shell is to a plate essentially as an arch is to a beam. As we saw in 6.2.2, the use of a nonlinear $P$-$M$ interaction locus makes the collapse analysis of arches difficult, and in practice a piecewise linear locus is needed. This simplification is all the more necessary for shells, in which more than one component of both membrane force and moment must in general be accounted for. As with one-dimensional members, a piecewise linear yield locus is generated by giving the shell an ideal sandwich structure. Alternatively, the piecewise linear locus may be viewed as an approximation to the “exact” nonlinear one for a solid (or nonideal sandwich) shell.

The geometry of a shell is usually described by its middle surface, analogous to the middle plane of a plate, so that the two free surfaces are at a distance $h/2$ from it. At any point of the middle surface, a local Cartesian basis $(e_i)$ may be formed such that $e_3$ is perpendicular to the middle surface, while $e_1$ and $e_2$ define its tangent plane. With respect to this basis, the stress resultants $N_{\alpha\beta}$, $Q_\alpha$, and $M_{\alpha\beta}$ may be defined in the same way as for plates. The equilibrium equations they satisfy are, of course, different; they require a global curvilinear coordinate system. The general theory of shells is not presented here; only some special cases are studied.

The deformation of the middle surface can be described by means of the strain tensor with components $\bar{\varepsilon}_{\alpha\beta}$, describing stretching, and the curvature tensor with components $\kappa_{\alpha\beta}$, describing bending. With shearing deformation neglected, the internal virtual work per unit surface area is

$$M_{\alpha\beta} \delta \kappa_{\alpha\beta} + N_{\alpha\beta} \delta \bar{\varepsilon}_{\alpha\beta},$$

and consequently, the $M_{\alpha\beta}$ and $N_{\alpha\beta}$ constitute the generalized stresses for the most general shearless theory of limit analysis of shells; the shear forces $Q_\alpha$ are reactions and do not enter the yield locus.

If the mechanical behavior of the shell is isotropic in the tangent plane, then the yield locus is expressible in terms of the principal moments $M_1$, $M_2$, and the principal membrane forces $N_1$, $N_2$. If, in addition, symmetry or another constraint require one of the principal strains $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$ or the principal curvatures $\kappa_1$, $\kappa_2$ to be zero, then the conjugate force or moment ceases to be a generalized force and becomes an internal reaction instead; it can therefore be eliminated from the yield locus, further reducing the number of dimensions of the space in which the yield locus must be described.

**Piecewise Linear Yield Locus**

Following Hodge [1959], we derive the piecewise linear yield locus on the basis of the ideal sandwich shell, made up of two thin sheets of thickness $t$ separated by a core of thickness $h$. The sheets are elastic–perfectly plastic and obey the Tresca yield criterion with a uniaxial yield stress $\sigma_Y$. The
principal stresses in the two sheets will be denoted \( \sigma^\pm_1, \sigma^\pm_2 \) and \( \sigma_1^-, \sigma_2^- \), respectively, the sign convention being chosen so that

\[
M_\alpha = \frac{1}{2}(\sigma^-_\alpha - \sigma^+_\alpha)ht, \quad \alpha = 1, 2,
\]

and

\[
N_\alpha = (\sigma^-_\alpha + \sigma^+_\alpha)t, \quad \alpha = 1, 2.
\]

Consequently,

\[
\sigma^\pm_\alpha = \frac{N_\alpha}{2t} \mp \frac{M_\alpha}{ht}.
\]

If the principal stresses \( \sigma^\pm_\alpha \) are not to violate the Tresca yield criterion, they must satisfy the six inequalities

\[
|\sigma^+_1| \leq \sigma_Y, \quad |\sigma^+_2| \leq \sigma_Y, \quad |\sigma^-_1 - \sigma^-_2| \leq \sigma_Y.
\]

In order to express these inequalities in terms of the \( M_\alpha \) and \( N_\alpha \), we define \( M_U = \sigma_Y ht \) and \( N_U = 2\sigma_Y t \), as well as the dimensionless quantities \( m_\alpha = M_\alpha/M_U, n_\alpha = N_\alpha/N_U \). We thus obtain

\[
|m_1 - n_1| \leq 1, \quad |m_1 + n_1| \leq 1,
\]

\[
|m_2 - n_2| \leq 1, \quad |m_2 + n_2| \leq 1,
\]

\[
|m_1 - m_2 + n_1 - n_2| \leq 1, \quad |m_1 - m_2 - n_1 + n_2| \leq 1.
\]

The six absolute-value inequalities (6.4.3) are equivalent to twelve algebraic inequalities, so that the yield locus is bounded by twelve hyperplanes in the four-dimensional \( m_1m_2n_1n_2 \)-space.

If one of the quantities \( m_\alpha, n_\alpha \) represents an internal reaction rather than a generalized stress, then it can be eliminated from the yield locus. Suppose this quantity to be \( m_2 \) (i.e., suppose the shell to be constrained so that \( \kappa_2 = 0 \)); then the inequalities involving \( m_2 \) may be rewritten as

\[
-1 + n_2 \leq m_2 \leq 1 + n_2, \quad -1 - n_2 \leq m_2 \leq 1 - n_2,
\]

\[
-1 + m_1 + n_1 - n_2 \leq m_2 \leq 1 + m_1 + n_1 - n_2,
\]

\[
-1 + m_1 - n_1 + n_2 \leq m_2 \leq 1 + m_1 - n_1 + n_2.
\]

The actual values of \( m_2 \) do not matter, as long as some \( m_2 \) can be found so that all the inequalities (6.4.4) can be satisfied, and this is the case whenever the first member of each inequality is no greater than the third member of every inequality. The first two inequalities give

\[
|n_2| \leq 1, \quad (6.4.5a)
\]

and the second two give

\[
|n_1 - n_2| \leq 1. \quad (6.4.5b)
\]
Combining inequalities from the first and second pairs leads to the following additional nontrivial inequalities:

\[
|2n_2 - n_1 + m_1| \leq 2, \quad |2n_2 - n_1 - m_1| \leq 2. \tag{6.4.5c-d}
\]

Together with (6.4.3), we thus have six absolute-value inequalities involving \(m_1, n_1\) and \(n_2\), or a total of twelve algebraic inequalities. The yield locus is therefore a dodecahedron in the three-dimensional \(m_1n_1n_2\)-space, illustrated in Figure 6.4.2. The derivation of this locus is due to Hodge [1954], who also derived the exact nonlinear yield locus for a solid shell made of uniform material. As discussed previously, the piecewise linear locus may be thought of as an approximation to the exact one when the appropriate values of \(M_U\) and \(N_U\) are used, namely \(M_U = \sigma_Y h^2/4\) and \(N_U = \sigma_Y h\), with \(\sigma_Y\) the uniaxial yield stress of the solid-shell material.

The plastic dissipation per unit area of the ideal sandwich shell, given rigid–plastic behavior, is

\[
\bar{D}_p = \frac{\sigma_Y t}{2} \left( |\dot{\varepsilon}_1^+| + |\dot{\varepsilon}_2^+| + |\dot{\varepsilon}_1^- + \dot{\varepsilon}_2^-| + |\dot{\varepsilon}_1^-| + |\dot{\varepsilon}_2^-| + |\dot{\varepsilon}_1^- + \dot{\varepsilon}_2^-| \right).
\]

But

\[
\dot{\varepsilon}_\alpha^\pm = \dot{\varepsilon}_\alpha \mp \frac{h}{2\kappa_\alpha}, \quad \alpha = 1, 2.
\]

Consequently,

\[
\bar{D}_p = \frac{1}{4} \left[ |N_U\dot{\varepsilon}_1 + M_U\dot{\kappa}_1| + |N_U\dot{\varepsilon}_1 - M_U\dot{\kappa}_1| \\
+ |N_U\dot{\varepsilon}_2 + M_U\dot{\kappa}_2| + |N_U\dot{\varepsilon}_2 - M_U\dot{\kappa}_2| \\
+ |N_U(\dot{\varepsilon}_1 + \dot{\varepsilon}_2) + M_U(\dot{\kappa}_1 + \dot{\kappa}_2)| + |N_U(\dot{\varepsilon}_1 + \dot{\varepsilon}_2) - M_U(\dot{\kappa}_1 + \dot{\kappa}_2)| \right]. \tag{6.4.6}
\]
Yield criteria that are appropriate for the approximate treatment of reinforced-concrete shells are discussed by Save and Massonnet [1972], Chapter 9.

6.4.3. Limit Analysis of Shells: Examples

Radially Loaded Cylindrical Shell: Basic Equations

As a first example of the application of the theory discussed above, we consider a circular cylindrical shell of uniform mean radius \( a \) and thickness \( h \), subject to an axisymmetric radial pressure distribution that may vary along the axial coordinate \( z \). Because of axial symmetry the displacement of the middle surface has only the radial component \( u(z) \) and the axial component \( w(z) \). The displacements throughout the shell are assumed to be governed by the “plane sections remain plane” hypothesis applied to a longitudinal strip subtending a small angle \( d\theta \) (see Figure 6.4.3):

\[
\begin{align*}
  u_r(r, z) &= u(z), & u_\theta(r, z) &= 0, \\
  u_z(r, z) &= w(z) - (r - a)u'(z).
\end{align*}
\]

In accordance with Equations (1.2.1), the only nonvanishing strain components are

\[
\varepsilon_\theta = \frac{u}{r} = \frac{u}{a}
\]

and

\[
\varepsilon_z = w' - yu''.
\]
where \( y = r - a \). The internal virtual work per unit area is
\[
\int_{-h/2}^{h/2} (\sigma_\theta \delta \varepsilon_\theta + \sigma_z \delta \varepsilon_z) \, dy = N_\theta \frac{\delta u}{a} + N_z \delta w' + M_z \delta u'',
\]
where
\[
N_\theta = \int_{-h/2}^{h/2} \sigma_\theta \, dy, \quad N_z = \int_{-h/2}^{h/2} \sigma_z \, dy, \quad M_z = -\int_{-h/2}^{h/2} y \sigma_z \, dy.
\]
The generalized stresses are thus three in number. The conjugate generalized strains are \( \bar{\varepsilon}_\theta = u/a \), \( \bar{\varepsilon}_z = w' \), and \( \kappa_z = u'' \). The fact that \( \kappa_\theta = 0 \) (i.e., circles remain circles) removes \( M_\theta \) from the rank of generalized stresses. The yield locus is therefore given by Equations (6.4.3)\textsubscript{1,2} and (6.4.5a)–(6.4.5d), with \( M_1 = M_z \), \( N_1 = N_z \), and \( N_2 = N_\theta \).

The plastic dissipation per unit area is
\[
\bar{D}_p = \frac{1}{4} \left[ 2N_U \frac{|\dot{u}|}{a} + |N_U \frac{\dot{w}'}{a} + M_U \dot{u}''| + |N_U \frac{\dot{w}'}{a} - M_U \dot{u}''| + |N_U \left( \frac{\dot{u}}{a} + \dot{w}' \right) + M_U \dot{u}''| \right]. \tag{6.4.7}
\]

The equilibrium equations can be obtained by applying the principle of virtual work. If the shell axis occupies the interval \(-L \leq z \leq L\), then the internal virtual work is
\[
\delta W_{\text{int}} = 2\pi a \int_{-L}^{L} (N_\theta \frac{\delta u}{a} + N_z \delta w' + M_z \delta u'') \, dz.
\]
Through integration by parts this becomes
\[
\delta W_{\text{int}} = 2\pi a \left\{ (M_z \delta u' - M_z' \delta u + N_z \delta w) \bigg|_{-L}^{L} \right. \\
+ \left. \int_{-L}^{L} \left[ (M_z'' + \frac{N_\theta}{a}) \delta u - N_z' \delta w \right] \, dz \right\}.
\]
In addition to the radial pressure \( p \) (positive outward), let the shell be loaded by a bending moment \( M_z^+ \), an axial force \( N_z^+ \), and a shear force \( Q_r^+ \), all per unit length of circumference, at \( z = L \), and similarly, \( M_z^- \), \( N_z^- \), and \( Q_r^- \) at \( z = -L \). The external virtual work is therefore
\[
\delta W_{\text{ext}} = 2\pi a \left[ \int_{-L}^{L} p \delta u \, dz + N_z^+ \delta w(L) + M_z^+ \delta u'(L) \right. \\
+ \left. Q_r^+ \delta u(L) - N_z^- \delta w(-L) - M_z^- \delta u'(-L) - Q_r^- \delta u(-L) \right].
\]
Equating the internal and external virtual work leads to the equilibrium equations
\[
N_z' = 0, \quad M_z'' + \frac{N_\theta}{a} = p \tag{6.4.8}
\]
and the boundary conditions

\[(M_z - M^\pm_z) \delta u' = 0, \quad (M'_z + Q^{\pm}_r) \delta u = 0, \quad (N_z - N^\pm_z) \delta w = 0, \quad z = \pm L.\]

Note that the axial end forces must be equal and opposite for equilibrium. If these are zero, then \(N_z = 0\) everywhere.

For the shell without end load, the piecewise linear yield locus reduces to a hexagon in the \(M_zN_{\theta}\)-plane, formed by the three pairs of parallel lines described by

\[|m| = 1, \quad |2n + m| = 2, \quad |2n - m| = 2, \quad (6.4.9)\]

where \(m = M_z/M_U\) and \(n = N_{\theta}/N_U\) (see Figure 6.4.4, which also shows the “exact” nonlinear yield locus as well as the simplified square locus given by \(|m| \leq 1, |n| \leq 1\).

![Yield loci for a cylindrical shell without end load.](image)

The plastic dissipation for the yield criterion described by (6.4.9) may be obtained from (6.4.7) by substituting, as in uniaxial stress, \(\dot{\varepsilon}_1 = -\frac{1}{2}\dot{\varepsilon}_2\), or \(\dot{u}' = -\dot{u}/2a\). The result is

\[\bar{D}_p = \frac{1}{2} \left( N_U \frac{|\dot{u}|}{a} + |N_U \frac{\dot{u}}{2a} + M_U \dot{u}'| + |N_U \frac{\dot{u}}{2a} - M_U \dot{u}'| \right). \quad (6.4.10)\]

When the state of generalized stress can be assumed to be represented by points on one of the inclined lines \(AB, BC, DE,\) and \(EF\) of Figure 6.4.4, then \(N_{\theta}\) can be expressed in terms of \(M_z\), and the result substituted in the equilibrium equation (6.4.8). The equation then becomes a linear differential equation for \(M_z\), which can be solved subject to appropriate boundary
conditions. Since the normality rule gives \(2M_U \dot{u}'' = \pm NU \dot{u}/a\) along the aforementioned lines, the dissipation is just \(NU |\dot{u}|/a\). The same result obtains at the vertices \(B\) and \(E\). Along the lines \(AF\) and \(CD\), normality requires \(\dot{u}/\dot{u}'' = 0\). A nonvanishing velocity thus implies \(\dot{u}'' = \pm \infty\), that is, the formation of a plastic hinge circle.

A simple case occurs when the pressure \(p\) is constant and the ends of the tube are free. Then \(M_z = 0\), and the yield criterion reduces to \(|N_\theta| = NU\). The ultimate pressure is therefore given by \(|p_U| = NU/a\).

A solution for a pressurized tube with clamped ends was derived by Hodge [1954]. Here the central portion of the tube is in regime \(DE\), and the outer portions in \(EF\); the boundary is at \(z = \pm \eta L\). The parameter \(\eta\) and the dimensionless ultimate pressure \(\bar{p} = p_U/a/NU\) are given implicitly as functions of the dimensionless shell parameter \(\omega\), defined by

\[
\omega^2 = \frac{NU L^2}{2M_U a}.
\]

The results are

\[
\sinh \omega \eta = \frac{\sin \omega (1 - \eta)}{\sqrt{2} \cos \omega (1 - \eta) + 1},
\]

\[
\bar{p} = \frac{2 - \cos \omega (1 - \eta)}{2[1 - \cos \omega (1 - \eta)]}
\]

for \(\omega \leq 1.65\), and

\[
\tan \omega (1 - \eta) = \coth \omega \eta,
\]

\[
\bar{p} = 1 + \frac{1}{2(2 \cosh \omega \eta - 1)}
\]

for \(\omega > 1.65\). For more details, see Hodge [1959], Section 11-2.

**Cylindrical Shell with a Ring Load**

As another example, suppose that a free-ended shell is subject to a radially inward ring loading at \(z = 0\), its intensity being \(F\) per unit length of circumference; that is,

\[
p(z) = -F \delta(z),
\]

where \(\delta(\cdot)\) is the Dirac delta function. Note that

\[
\delta(z) = \frac{1}{2} \frac{d}{dz} \text{sgn } z = \frac{1}{2} \frac{d^2}{dz^2} |z|.
\]

It is reasonable to assume initially that the hoop stress is compressive, that is, \(N_\theta \leq 0\). If the shell is extremely short, then its collapse should not depend very much on whether the force is applied around a ring or uniformly distributed over the surface in a statically equivalent manner. For very short
shells, then, we should expect \( F_U = 2N_U L/a \), with bending having little or no effect.

When bending is taken into account, it can be seen that at least in a central portion of the tube, \( M_z \geq 0 \). It will be assumed, to begin with, that the entire shell is plastic and in regime \( AB \) of Figure 6.4.4. Eliminating \( N_\theta \), we obtain the following dimensionless differential equation for \( m(\zeta) \), using the dimensionless variable \( \zeta = z/L \), and the parameters \( f = Fa/(2N_U L) \) and \( \omega \) as previously defined:

\[
m''(\zeta) + \omega^2 m(\zeta) = 2\omega^2 - 4\omega^2 f \delta(\zeta).
\]

The general solution of this equation that is even in \( \zeta \) is

\[
m(\zeta) = 2 - 2\omega f \sin(\omega |\zeta|) + C \cos \omega \zeta,
\]

with \( C \) an arbitrary constant. The free-end condition \( m(1) = m'(1) = 0 \) yields \( C = -2 \cos \omega \) and

\[
f = \frac{\sin \omega}{\omega}. \tag{6.4.11}
\]

The solution may accordingly be written as

\[
m(\zeta) = 2[1 - \cos \omega(1 - |\zeta|)].
\]

The requirement that \( 0 \leq m \leq 1 \) limits the validity of this solution to \( \omega \leq \pi/3 \). For sufficiently short shells, then, Equation (6.4.11) gives a lower bound to the collapse load; in particular, the limit as \( \omega \to 0 \) is \( f = 1 \), as previously discussed. An associated kinematically admissible velocity field, however, can easily be found, namely,

\[
\dot{u}(\zeta) = -v_0 \cos \omega \zeta. \tag{6.4.12}
\]

It can readily be checked that the generalized strain rates derived from this velocity field satisfy the normality condition for regime \( AB \), and hence Equation (6.4.11) gives the collapse load for \( \omega \leq \pi/3 \).

Alternatively, the upper-bound theorem can be applied directly to the velocity field (6.4.12), the dissipation per unit length being \( 2\pi N_U |\dot{u}| \), and the external work rate \( 2\pi a F v_0 \). The result is Equation (6.4.11) as an upper bound, as long as \( \dot{u}(\zeta) \) does not change sign; this condition is met for \( \omega \leq \pi/2 \). Equation (6.4.11) is therefore an upper bound for \( \pi/3 \leq \omega \leq \pi/2 \). However, a better upper bound can be obtained for this range by assuming a velocity field with a plastic hinge circle at \( \zeta = 0 \), namely,

\[
\dot{u}(\zeta) = -v_0 (\cos \omega \zeta + \beta \sin \omega |\zeta|). \tag{6.4.13}
\]

The additional dissipation at the hinge circle is \( 4\pi a M_U \beta v_0 / L = 2\pi N_U L \beta v_0 / \omega^2 \). Equating dissipation and external work rate yields

\[
f = \sin \omega - \frac{\beta}{2\omega^2} (1 - 2 \cos \omega)
\]

The general solution of this equation that is even in \( \zeta \) is

\[
m(\zeta) = 2 - 2\omega f \sin(\omega |\zeta|) + C \cos \omega \zeta,
\]

with \( C \) an arbitrary constant. The free-end condition \( m(1) = m'(1) = 0 \) yields \( C = -2 \cos \omega \) and

\[
f = \frac{\sin \omega}{\omega}. \tag{6.4.11}
\]

The solution may accordingly be written as

\[
m(\zeta) = 2[1 - \cos \omega(1 - |\zeta|)].
\]

The requirement that \( 0 \leq m \leq 1 \) limits the validity of this solution to \( \omega \leq \pi/3 \). For sufficiently short shells, then, Equation (6.4.11) gives a lower bound to the collapse load; in particular, the limit as \( \omega \to 0 \) is \( f = 1 \), as previously discussed. An associated kinematically admissible velocity field, however, can easily be found, namely,

\[
\dot{u}(\zeta) = -v_0 \cos \omega \zeta. \tag{6.4.12}
\]

It can readily be checked that the generalized strain rates derived from this velocity field satisfy the normality condition for regime \( AB \), and hence Equation (6.4.11) gives the collapse load for \( \omega \leq \pi/3 \).

Alternatively, the upper-bound theorem can be applied directly to the velocity field (6.4.12), the dissipation per unit length being \( 2\pi N_U |\dot{u}| \), and the external work rate \( 2\pi a F v_0 \). The result is Equation (6.4.11) as an upper bound, as long as \( \dot{u}(\zeta) \) does not change sign; this condition is met for \( \omega \leq \pi/2 \). Equation (6.4.11) is therefore an upper bound for \( \pi/3 \leq \omega \leq \pi/2 \). However, a better upper bound can be obtained for this range by assuming a velocity field with a plastic hinge circle at \( \zeta = 0 \), namely,

\[
\dot{u}(\zeta) = -v_0 (\cos \omega \zeta + \beta \sin \omega |\zeta|). \tag{6.4.13}
\]

The additional dissipation at the hinge circle is \( 4\pi a M_U \beta v_0 / L = 2\pi N_U L \beta v_0 / \omega^2 \). Equating dissipation and external work rate yields

\[
f = \sin \omega - \frac{\beta}{2\omega^2} (1 - 2 \cos \omega)
\]
if \( \dot{u} \) does not change sign, that is, if \( \beta \leq \omega \cot \omega \). Choosing this limiting value for \( \beta \) gives the smallest value of \( f \), namely,

\[
f = \frac{2 - \cos \omega}{2 \omega \sin \omega},
\]

which is less than the right-hand side of (6.4.11) for \( \pi/3 < \omega < \pi/2 \).

A lower bound for \( \omega > \pi/3 \) can be obtained by assuming the solution

\[
m(\zeta) = 2 - \cos \omega \zeta - 2\omega f \sin(\omega |\zeta|)
\]

valid for \( |\zeta| < \eta \), where \( \eta \) is such that \( m(\eta) = m'(\eta) = 0 \), and continuing it statically as \( M_z = 0, N_\theta = 0 \) for \( |\zeta| > \eta \). The conditions at \( \eta \) lead to \( \omega \eta = \pi/3 \) and \( f = \sqrt{3}/(2\omega) \), or, in dimensional form,

\[
F = \sqrt{\frac{6M_U N_U}{a}},
\]

a result that can be seen to be independent of the length. As can be seen, however, as the length increases, so does the discrepancy between the upper and lower bounds.

For longer shells the solution based on the hexagonal yield locus becomes difficult. For an infinitely long shell, Drucker [1954b] found the collapse load

\[
F = 2\sqrt{\frac{3M_U N_U}{a}} \tag{6.4.14}
\]

or \( f = \sqrt{3}/2/\omega \). This load is based on the following moment distribution:

\[
m(\zeta) = 2 - \cos \omega \zeta - \frac{\sqrt{6}}{\omega} \sin \omega |\zeta|, \quad 0 \leq |\zeta| \leq \zeta_1 \quad (AB),
\]

\[
= -2 + \cosh \omega (|\zeta| - \zeta_2), \quad \zeta_1 \leq |\zeta| \leq \zeta_3 \quad (BC),
\]

\[
= -2 + 2 \cos \omega (|\zeta| - \zeta_4), \quad \zeta_3 \leq |\zeta| \leq \zeta_4 \quad (DE),
\]

\[
= 0, \quad \zeta_4 \leq |\zeta| \quad \text{(rigid)},
\]

where

\[
\omega \zeta_1 = \cos^{-1} \left( \frac{2 + 3\sqrt{2}}{7} \right) = 0.469, \quad \omega(\zeta_2 - \zeta_1) = \cosh^{-1} 2 = 1.317,
\]

\[
\omega(\zeta_3 - \zeta_2) = \frac{1}{2} \cosh^{-1} 4 = 1.032, \quad \omega(\zeta_4 - \zeta_3) = \frac{1}{2} \cosh^{-1} \frac{1}{4} = 0.659.
\]

It follows that \( \omega \zeta_4 = 3.477 \), and therefore the result is valid for \( \omega \geq 3.477 \). Note that \( m(0) = 1 \) and \( m(\zeta_2) = -1 \), so that plastic hinges develop at those points. At \( \zeta_3 \) the hoop stress \( N_\theta \) changes abruptly from a negative to a positive value, and at \( \zeta_4 \) back to zero.
As pointed out by Drucker [1954b], the mathematics is greatly simplified if the yield locus is replaced by the rectangle $|M_z| \leq M_U$, $|N_\theta| \leq N_U$. With this yield criterion, Eason and Shield [1955] found complete solutions for shells of all lengths, and with the load not necessarily at the center. Since the rectangle circumscribes the hexagon, the collapse loads found by Eason and Shield are upper bounds on those that would be found for the hexagonal yield locus. Furthermore, a rectangle with vertices at $(\pm M'_U, \pm N'_U)$ may be inscribed in the hexagon; then the collapse load for the rectangle is a lower bound for the hexagon when $M_U$ and $N_U$ are respectively replaced by $M'_U$ and $N'_U$. The values of $M'_U$ and $N'_U$ may be chosen to as to maximize the lower bound.

With the rectangular yield locus, both kinematic and static solutions are quite easy. The sides $|M_z| = M_U$ of the rectangle (like those of the hexagon) correspond to hinge circles, while the sides $|N_\theta| = N_U$ describe velocities $\dot{u}$ varying linearly with position. For each side, $M_z$ and $N_\theta$ are polynomial functions of $z$.

For the symmetric problem under consideration, Eason and Shield’s results are

$$f = \frac{1}{2} \left( \frac{1}{\omega^2} + 1 \right), \quad \omega \leq 1 + \sqrt{2},$$

$$f = \frac{\sqrt{2}}{\omega}, \quad \omega \geq 1 + \sqrt{2}.$$

The latter result is equivalent to

$$F = 4 \sqrt{\frac{M_U N_U}{a}}, \quad (6.4.15)$$

independent of the length. Equation (6.4.15) also gives the lower bound

$$F = 4 \sqrt{\frac{M'_U N'_U}{a}},$$

where

$$\frac{M'_U}{M_U} + 2 \frac{N'_U}{N_U} \leq 2, \quad M'_U \leq M_U.$$

The lower bound is maximized for $M'_U = M_U$, $N'_U = \frac{1}{2} N_U$, and equals $2 \sqrt{2 M_U N_U / a}$. The bounds thus bracket Drucker’s collapse load (6.4.14) for the long shell.

**Spherical Cap Under Pressure**

In shells of revolution, as a rule, all four generalized stresses are active. An approximate theory, in which the effects of one of them are ignored, was proposed by Drucker and Shield [1959], but its applicability is limited.

It was shown by Hodge [1959] that for a shell of revolution, a point on the four-dimensional piecewise linear yield locus (6.4.3) corresponds to
plastic deformation only if it lies on the intersection of two of the twelve hyperplanes, one of which represents yielding of the top sheet and the other represents yielding of the bottom sheet. The generalized stresses thus satisfy two yield equations and two equilibrium equations, making the problem “statically determinate.” Furthermore, there are two normality conditions on the generalized strain rates, and since the latter are derived from two velocity components, the problem is “kinematically determinate” as well.

We consider here a spherical cap of radius $a$ subtending a half-angle $\phi_0$, clamped around its edge, and carrying a uniform radial pressure $p$ (see Figure 6.4.5). The problem was treated by Onat and Prager [1954] on the basis of a nonlinear yield criterion and by Hodge [1959, Section 11-6] on the basis of the piecewise linear yield criterion (6.4.3).

Figure 6.4.5. Spherical cap under uniform external pressure: geometry and loading.

If the radial displacement is $u_r = u$ and the meridional displacement is $u_\phi = v$, then the generalized strains are

$$\bar{\varepsilon}_\theta = \frac{u + v \cot \phi}{a}, \quad \bar{\varepsilon}_\phi = \frac{u + v'}{a},$$

$$\kappa_\theta = \cot \phi \frac{u' - v}{a^2}, \quad \kappa_\phi = \frac{u'' - v'}{a^2},$$

where $(\cdot)' = d(\cdot)/d\phi$.

To find an upper bound on the collapse pressure, we assume the velocity field

$$\dot{u} = -v_0(\cos \phi - \cos \phi_0), \quad \dot{v} = 0.$$

The generalized strain rates are accordingly

$$\dot{\bar{\varepsilon}}_\theta = \dot{\bar{\varepsilon}}_\phi = -\frac{v_0}{a} (\cos \phi - \cos \phi_0), \quad \dot{\kappa}_\theta = \dot{\kappa}_\phi = \frac{v_0}{a^2} \cos \phi.$$

The plastic dissipation per unit area is then, from Equation (6.4.6),

$$\bar{D}_p = NUv_0[|\cos \phi_0 - (1 - k) \cos \phi| + |(1 + k) \cos \phi - \cos \phi_0|]$$
\[
= \begin{cases} 
2N_U v_0 (\cos \phi - \cos \phi_0), & 0 \leq \phi \leq \phi^*, \\
2k N_U v_0 \cos \phi, & \phi^* \leq \phi \leq \phi_0,
\end{cases}
\]

where
\[
k = \frac{M_U}{N_U a}, \quad \cos \phi^* = \frac{\cos \phi_0}{1 - k}
\]

The expression for \( \phi > \phi^* \) is necessary, of course, only if \( \phi_0 > \phi^* \).

In addition, since \( \dot{u}' \) does not vanish at the edge \( \phi = \phi_0 \), a plastic hinge circle forms there, necessitating the additional dissipation (per unit length) given by \( M_U v_0 \sin \phi_0 / a = k N_U v_0 \sin \phi_0 \). The total internal dissipation is thus
\[
\mathcal{D}_{\text{int}} = 2\pi a^2 \int_0^{\phi_0} \bar{D}_p \sin \phi \, d\phi + 2\pi a^2 k N_U v_0 \sin^2 \phi_0.
\]

The external work rate is
\[
\mathcal{D}_{\text{ext}} = 2\pi a^2 \int_0^{\phi_0} p \dot{u} \sin \phi \, d\phi = \pi a^2 p v_0 (1 - \cos \phi_0)^2.
\]

Setting \( \mathcal{D}_{\text{ext}} = \mathcal{D}_{\text{int}} \) yields the upper bound
\[
p = 2 \frac{N_U}{a} \left[ 1 + k \frac{1 + \cos \phi_0}{1 - \cos \phi_0} + \frac{k^2}{1 - k} \left( \frac{\cos \phi_0}{1 - \cos \phi_0} \right)^2 \right], \quad \cos \phi_0 \leq 1 - k,
\]
\[
p = 4k \frac{N_U}{a} \frac{1 + \cos \phi_0}{1 - \cos \phi_0}, \quad \cos \phi_0 \geq 1 - k.
\]

A lower bound may be obtained by assuming that the stress field in the shell is one of simple membrane compression, \( N_\theta = N_\phi = -N_U \), with \( M_\theta = M_\phi = 0 \). The corresponding lower-bound pressure is \( p = 2N_U / a \).

A lower bound that is better for sufficiently small cap angles was found by Hodge [1959] by assuming \( N_\theta, N_\phi, M_\theta \) and \( M_\phi \) to be sinusoidally varying functions of \( \phi \), substituting into the equilibrium equations, and choosing the free coefficients so as to maximize the pressure subject to the yield inequalities. The resulting best lower bound is given by
\[
\frac{pa}{N_U} = 2 + \frac{1}{1 - \cos \phi_0} \left[ 1 - \sqrt{\left( \frac{1 - k}{1 + k} \right)^2 + 4 \left( \frac{1 - \cos \phi_0}{1 + \cos \phi_0} \right)^2} \right].
\]

Clearly, this result is an improvement over the previous one if and only if the quantity under the square-root sign is less than unity.

**Exercises: Section 6.4**

1. Using yield-line theory, find upper bounds for a uniformly loaded, simply supported Tresca plate having the shape of (a) an equilateral triangle, (b) a right isosceles triangle, and (c) a regular hexagon.
2. Repeat Exercise 1 for clamped plates.

3. Find lower bounds for some of the plates in Exercises 1 and 2.

4. Using yield-line theory, find upper bounds for the plates of Exercises 1 and 2 when they carry a single concentrated load at the center or centroid. Compare with the result \( F_U = 2\pi M_U \).

5. A square Tresca plate carrying a uniform downward load is simply supported along its edges against downward deflection but is free to lift off. Using yield-line theory, find an upper bound to the collapse load.

6. Using some of the methods of Section 6.2, find lower and upper bounds for the yield locus of an ideal sandwich shell obeying the Mises criterion and its associated flow rule (a) when \( M_2 \) is not a generalized stress and (b) when, in addition, \( N_1 = 0 \).

7. Find the complete solution for the clamped pressurized cylindrical shell with clamped ends obeying the “square” yield locus of Figure 6.4.4. Plot the nondimensional ultimate pressure \( \bar{p} = p_U a / N_U \) against the shell parameter \( \omega \).

8. Repeat Exercise 7 for a shell that is (a) clamped at one end and free at the other and (b) simply supported at both ends.

9. Repeat Exercise 8 for the hexagonal yield locus.

10. For a short free-ended cylindrical shell subject to a noncentered ring load, find the collapse load based on the “square” yield locus.