Asymptotic Formulae for Some One Dimensional Network Flow Problems

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1. Introduction

Asymptotic formulae exist for the traveling salesman problem, "TSP", (Eilon et al., 1971, Karp, 1977, Daganzo, 1984a), and the vehicle routing problem, "VRP", (Eilon et al., 1971, Daganzo, 1984b, Haimovich et al., 1985, Newell and Daganzo, 1986 and 1986a). The results apply to problems with N randomly and homogeneously distributed points on a region of a metric plane with area A, and density $\delta = N/A$. In all cases the distance traveled per point for the TSP, or the "detour" distance per point for the VRP, tends to a fixed multiple of the characteristic separation between points as N and A increase in a fixed ratio. This limiting result holds for problems defined in K-dimensional regions. This paper presents related but qualitatively different results for single-commodity network flow problems with random inputs on undirected paths.

2. Definitions

Consider the transportation problem of linear programming, "TLP". Given are N points, a set of non-negative inter-point distances, $\{d_{ij}, \forall i,j=1..N \text{ with } i\neq j\}$, satisfying the triangle inequality, and a set of net point supplies, $\{v_i\}$, measured in items. Positive v_i are interpreted as supplies and negative values as demands. The goal is to find a set of non-negative shipments, $\{v_{ij}, \forall i,j=1..N \text{ with } i\neq j\}$, that minimize the total distance traveled and satisfy supply and demand constraints.

(TLP) min $$z = \sum_{ij} d_{ij} v_{ij} \quad (1a)$$

s. t.: $$\sum_j (v_{ij} - v_{ji}) \le v_i \;;\quad \forall i = 1..N \quad (1b)$$

$$v_{ij} \ge 0 \;;\quad \forall i = 1..N,\ j = 1..N; i \neq j\ . \quad (1c)$$

Conservation equations (1b) ensure that the net flow emanating from a point i never exceeds the net supply. For $v_i < 0$, the net flow into i must exceed or equal the positive net demand at i. Note eqs.(1) do not differentiate explicitly between sinks and sources, and do not rule out multi-link routes. However, if the $\{d_{ij}\}$ satisfy the triangle inequality, multi-link routes cannot be optimal. In particular the optimal solution cannot have inflow into a source or outflow from a sink.

Problem TLP is feasible if $\sum_i v_i \ge 0$ and balanced, denoted TLP(B), if $\sum_i v_i = 0$. We define an always-feasible auxiliary problem, ATLP, that includes a fictitious source, $i=0$, with non-negative net supply, $v_0 = \left(-\sum_i v_i\right)^+$, and distances, $d_{i0}, d_{0j} = M >> \sup(d_{ij})$, representing a penalty for items not sent. The optimal cost of the ATLP, z^*, includes a distance component corresponding to real points, d^*, and a penalty component for the fictitious source. The penalty component, d^*-z^*, is v_0M, since (i) for a feasible TLP, TLP ≡ ATLP, $d^*= z^*$ and $v_0 = 0$; and (ii) for an infeasible TLP, ATLP is a balanced TLP, with fictitious outflow v_0. Thus, $d^* = z^* - v_0 M$.

The DTLP (or "depot-TLP") is a variant of the ATLP, where excess supplies are sent to the extra point, or depot; i.e., it is an ordinary TLP, with the net supply at the depot balancing the problem. The depot distances need not be large but must be non-negative, and must satisfy the triangle inequality. The minimum of the DTLP, z_D, is denoted d_D*. The DTLP can be used as a tool to develop upper bounds for d^* in higher dimensions. It is shown in Daganzo and Smilowitz

(2000) that for any TLP data, any associated DTLP satisfies $d_D^* \geq d^*$. For a balanced TLP, $d^* = d_D^*$. These relationships should be intuitive since the DTLP entails the extra effort to carry unused supplies to the depot, and when the TLP is balanced there are no unused supplies.

3. Formulae

This section develops formulae for the averages of d^* and d_D^* over a set of solutions when conditions vary. Points are embedded in the normed linear space R^1, and are identified by a single coordinate, x_i. Distance is $d_{ij} = \left| x_i - x_j \right|$. These conditions include TLP and DTLP problems on undirected paths with link costs $b_{ji}=b_{ij} \geq 0, \forall (i,j) \ni \left| i-j \right| = 1$. We define a curve of cumulative supply vs. position, $v(x) = \sum_{x_i \leq x} v_i$, as in Fig. 1a, and consider the TLP(B).

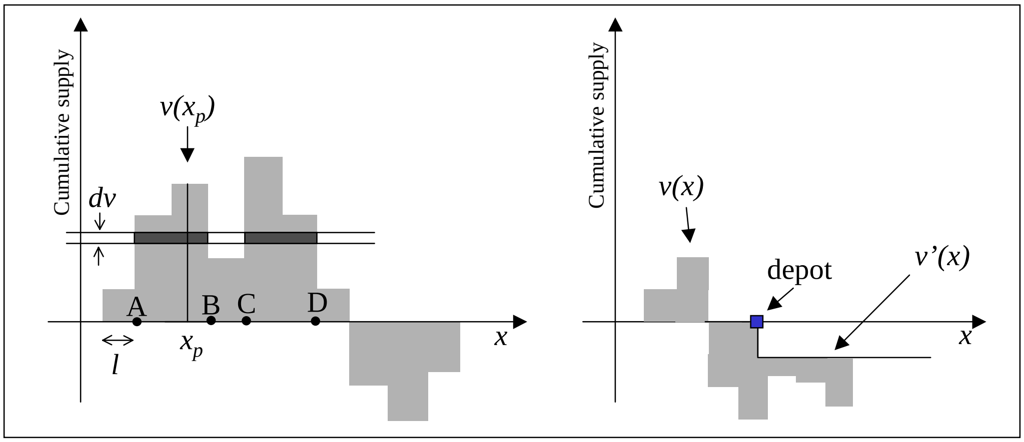

Figure 1. Graphical solutions of 1-D problems (a) TLP(B); (b) DTLP(U)

Result 1. (Formula for TLP(B)). The minimum cost for TLP(B) is the absolute area between *v(x)* and the x-axis; i.e.,

$$d^* = \int_{-\infty}^{+\infty} \left| v(x) \right| dx = \sum_{i=1}^{N-1} \left| v(x_i) \right| \left| x_{i+1} - x_i \right| . \blacksquare \tag{2}$$

Proof: It suffices to show that (2) is both an upper bound and a lower bound for d^*. For any point, x_p, such as the one in Fig. 1a, the net flow across x_p in any feasible solution must be $v(x_p)$ because the aggregate supply and demand on both sides of x_p must be satisfied. Thus, $\left| v(x_p) \right| dx$ is a lower bound to the optimal distance traveled in any small interval, (x_p, $x_p + dx$) where *v(x)* is constant. Therefore, the sum on the right side of (2) is a lower bound for d^*.

Conversely, a feasible solution can be constructed by considering horizontal slices of *dv* items (as in the figure) and transporting these quantities from points where the slice intersects a rising portion of curve *v(x)* to adjoining points where it intersects a falling portion. In the figure, *dv* items would be carried from A to B and from C to D. Thus, the summation of all the slices for small *dv* (still given by (2)) is the distance of a feasible solution and an upper bound to d^*. ■

If points are on a grid with lattice spacing *l*, then (2) reduces to $d^* = l \sum_{i=1}^{N-1} \left| v(x_i) \right|$. The notation TLP(B,G) will be used for this type of problem ("G" for grid). Let us examine the expected value of d^* for TLP(B,G) with random net supplies. For any homogeneous, balanced problem the net supplies must have zero mean, $\langle v_i \rangle = 0$, the same absolute mean $\langle |v_i| \rangle = \mu$, and the same variance, $\rangle v_i \langle = \sigma^2$. The (equal) covariances must be $\langle v_i v_i \rangle = -\sigma^2 /(N-1)$ since this covariance ensures that the variance of the sum of all the net supplies is zero. The general formulae for the mean, absolute mean, and variance of the sum of the first *i* net supplies, *v(x$_i$)*, are then:

$$\langle v(x_i) \rangle = 0\,, \quad \langle |v(x_i)| \rangle = i\mu\,, \text{ and} \quad \rangle v(x_i) \langle \; = \; i\sigma^2 \left[1 - \frac{i-1}{N-1} \right] \tag{3}$$

Assume now that the v_i have a joint multinormal distribution, and recall that if *X* is a normal random variable with zero-mean,

$$\langle |X| \rangle = [2 \rangle X \langle / \pi]^{1/2} \quad . \tag{4}$$

Thus, in the multinormal case, we have:

$$\langle |v(x_i)| \rangle = \left(2/\pi \right)^{1/2} \sigma \left(i \left[1 - \frac{i-1}{N-1} \right] \right)^{1/2} . \tag{5}$$

Result 2. (Expected optimal cost of TLP(B,G)). For the homogeneous, zero-mean TLP(B,G) with normal demand in R^1,

$$\langle d^*_{BG} \rangle = \left(2/\pi \right)^{1/2} \sigma l \sum_{i=1}^{N-1} \left(i \left[1 - \frac{i-1}{N-1} \right] \right)^{1/2} . \tag{6a}$$

Furthermore, the limit of this expression for $N \to \infty$ is

$$\langle d^*_{BG} \rangle \to \left(\frac{\pi}{32} \right)^{\frac{1}{2}} \sigma l N^{\frac{3}{2}} \; . \; \blacksquare \tag{6b}$$

Equation (6b) is true because the right hand side of (6a) is a Riemann sum that becomes a definite integral for $N \to \infty$, and such integral reduces to (6b). The details are as follows:

$$\left(2/\pi \right)^{1/2} \sigma l \int_0^N \left[x \left(1 - \frac{x}{N} \right) \right]^{\frac{1}{2}} dx = \left(2/\pi \right)^{1/2} \sigma l N^{\frac{3}{2}} \int_0^1 \left[\left(\frac{x}{N} \right) \left(1 - \frac{x}{N} \right) \right]^{\frac{1}{2}} d\left(\frac{x}{N} \right) = \left(\frac{\pi}{32} \right)^{\frac{1}{2}} \sigma l N^{\frac{3}{2}} . \tag{7}$$

In terms of the point density, $\delta = 1/l$, the average distance per point, $\langle p^*_{BG} \rangle = \langle d^*_{BG} \rangle / N$, is:

$$\langle p^*_{BG} \rangle \to \sqrt{\frac{\pi}{32}} \sigma \delta^{-1} \sqrt{N} \; . \tag{8}$$

This function increases without limit with the number of points, unlike in the TSP, where the average distance per point tends to a limit. The dependence with $\sqrt{N}$ is caused by the long-range interactions arising from the flow balancing requirements.

Equations (6b and 8) are quite general. It is shown below that they hold if the v_i are not normal, but satisfy the regularity conditions of Cramér's version of the central limit theorem for large deviations. They hold too if point locations vary in a segment of length *L* as a homogeneous Poisson process with rate δ. Here, *N* is the average number of points, $N = \delta L$.

Non-normality: Let $F_{v(x)}$ be the c.d.f. of *v*(*x*) for a given *N* and let $N \to \infty$ with a fixed interval length, *L*. Then, *v*(*x*) is the sum of $xN/L \to \infty$ net supplies, and the central limit theorem applies. Recall that $\langle d^* \rangle = \int_0^L \langle |v(x)| \rangle dx$, and let Φ be the c.d.f. of the normal approximation to *v*(*x*). Note now that $\left\langle |v(x)| \right\rangle = \int_{-\infty}^{0} \mathrm{F}_{v(x)}(z)dz + \int_{0}^{\infty} [1 - \mathrm{F}_{v(x)}(z)]dz \approx \int_{-\infty}^{0} \Phi(z)dz + \int_{0}^{\infty} [1 - \Phi(z)]dz$. The central limit theorem for large deviations (see corollary on p.553 of Feller, 1971) guarantees that the error in $\langle |v(x)| \rangle$ can be made smaller than any $\varepsilon > 0$ for all $v(x)$. Thus, the error in $\langle d^* \rangle$ is bounded by εL . QED

Random-point locations: Since $v(x)$ is a compound Poisson random variable, the central limit theorem still applies and $\langle |v(x)| \rangle \rightarrow [2\langle\rangle v(x)\langle\rangle / \pi]^{1/2}$. It suffices to show $\rangle v(x) \langle \rightarrow \sigma^2 N\left[\frac{x}{L}\left(1 - \frac{x}{L}\right)\right]$ because then the expectation of (2), $\langle d^* \rangle = \int_{-\infty}^{+\infty} \langle |v(x)| \rangle dx$, is (7). To establish this result, let *i(x)* be the number of points in [0, *x*], and note the conditional random variable ($v(x)|i(x)$) has zero mean, and variance $\rangle v(x) \mid i(x) \langle = i(x)\sigma^2 \left(1 - \frac{(i(x) - 1)}{(N-1)}\right)$. For zero mean, the unconditional variance is the expectation of the conditional variance; i.e. $\rangle v(x) \langle \; = \sigma^2 \left[\frac{\langle i(x) \rangle N}{N-1} - \frac{\langle i(x)^2 \rangle}{N-1}\right] = \sigma^2 \left[\frac{N^2 x}{L(N-1)} - \frac{1}{N-1}\left[\left(\frac{Nx}{L}\right)^2 + N\left(\frac{x}{L}\right)\left(1 - \frac{x}{L}\right)\right]\right] \rightarrow \sigma^2 N\left[\frac{x}{L}\left(1 - \frac{x}{L}\right)\right]$. QED

Unbalanced problems: An exact expression for TLP(U) is more difficult to obtain, yet it is shown in Daganzo and Smilowitz (2001) that $|\langle p^*_{BG} \rangle - \langle p^*_{UG} \rangle|$ = O($N^{1/2}$), implying that $\langle p^*_{UG} \rangle$ is also O($N^{1/2}$) for unbalanced problems. Equation (8) also holds for DTLP(B) since DTLP(B) ≡ TLP(B). The expression for DTLP(U) is different, but qualitatively similar. To derive it, first define the cumulative demand for the depot *v'(x)* as shown in Fig. 1b; i.e., $v'(x) = -v_0 H(x - x_0)$, where *H* is the Heaviside unit step function and x_0 is the depot location. Then, the same arguments used with Result 1 establish that the absolute area between curves *v* and *v'* is:

<u>Result 3. (Deterministic DTLP).</u> For both grid and random problems,

$$d^*_D = \int_{-\infty}^{+\infty} |v(x) - v'(x)| dx \;. \blacksquare \qquad (9)$$

For the DTLP(U), the expectation of the integrand of (9) is symmetric with respect to the location of the depot. Thus, for an integration region [0, *L*] it can be simplified as follows:

$$\langle d^*_{UD} \rangle = \int_0^L \langle |v(x) - v'(x)| \rangle dx = 2\int_0^{L/2} \langle |v(x) - v'(x)| \rangle dx = 2\int_0^{L/2} \langle |v(x)| \rangle dx$$, which from (4) is

$$\langle d^*_{UD} \rangle = 2\int_0^{L/2} \langle |v(x)| \rangle dx \rightarrow 2\int_0^{L/2} [(2/\pi)\sigma^2 (Nx/L)]^{1/2} dx = \sqrt{\frac{4}{9\pi}} \sigma\, LN^{1/2}$$, i.e., we have

<u>Result 4. (Expected optimal cost of DTLP(U)).</u>

$$\left\langle d^*_{UD} \right\rangle \rightarrow \sqrt{\frac{4}{9\pi}} \sigma\, lN^{\frac{3}{2}}, \text{ and } \quad \left\langle p^*_{UD} \right\rangle \rightarrow \sqrt{\frac{4}{9\pi}} \sigma\, lN^{\frac{1}{2}} \;. \blacksquare \qquad (10)$$

Note that in all cases, $\langle d^*\rangle = O(N^{3/2})$ when one holds l constant as N is increased. However, $\langle d^*\rangle = O(N^{1/2})$ if one holds the total region size, $L = lN$, constant.

The form of convergence of d^* is examined next. Unlike the TSP d^*_{UD} does not converge in probability to (10) because its coefficient of variation tends to a positive constant as $N\rightarrow\infty$.

Result 5. (Asymptotic coefficient of variation of the optimal distance for DTLP(U))

The asymptotic coefficient of variation tends to a positive constant as N increases; i.e., $\rangle d^*_{UD}\langle^{1/2} / \langle d^*_{UD}\rangle \rightarrow$ (constant) > 0. ∎

Proof: Recall d^*_{UD} is a Riemann sum of the distance in small intervals of [0, L], with a second moment, $\langle {d^*_{UD}}^2\rangle = \langle \int_0^L\int_0^L |v(x)-v'(x)||v(y)-v'(y)|dxdy\rangle = \int_0^L\int_0^L \langle |v(x)-v'(x)||v(y)-v'(y)|\rangle dxdy$.

By symmetry, $\langle {d^*_{UD}}^2\rangle = 2\int_0^{L/2}\int_0^{L/2}\langle | v(x)||v(y)|\rangle dxdy \ + 2\int_0^{L/2}\left(\int_{L/2}^{L}\langle |(v(x)|| [v(y)-v'(y)]|\rangle dy\right)dx$.

For large N, curve $v(x)$ has independent increments. Thus, the integrand of the second integral involves the product of independent quantities since the absolute net supplies being multiplied correspond to non-overlapping regions of [0, L]. The expression can be rewritten using the product of the expectations as:

$$\langle {d^*_{UD}}^2\rangle = 2\int_0^{L/2}\int_0^{L/2}\langle | v(x)||v(y)|\rangle dxdy \ + 2\int_0^{L/2}\left(\int_{L/2}^{L}\langle |(v(x)|\rangle\langle | [v(y)-v'(y)]|\rangle dy\right)dx$$

$$= 2\int_0^{L/2}\int_0^{L/2}\langle | v(x)||v(y)|\rangle dxdy \ + 2\int_0^{L/2}\int_0^{L/2}\langle |(v(x)|\rangle \langle |v(y)|\rangle dxdy \ .$$

The second inequality follows from symmetry. Note now that the first integrand satisfies:

$\langle | v(x)||v(y)|\rangle = \langle | v(x)|\rangle \langle |v(y)|\rangle + \text{cov}\{ | v(x)|, | v(y) |\}$. Thus,

$$\langle {d^*_{UD}}^2\rangle = 4\int_0^{L/2}\int_0^{L/2}\langle |(v(x)|\rangle \langle |v(y)|\rangle dxdy \ + 2\int_0^{L/2}\int_0^{L/2}\text{cov}(|v(x)|, |v(y)|)dxdy$$

The covariance integrand is strictly positive for all x and y not equal to 0, because $v(x)$ and $v(y)$ share the net supplies from 0 to $min(x, y)$. An expression for $\text{cov}(|v(x)|, |v(y)|)$ is of the form $\sigma^2 N\,\rho(x/L, y/L)$, and ρ is positive if both its arguments are positive. The second moment is then:

$$\langle {d^*_{UD}}^2\rangle = 4\int_0^{L/2}\int_0^{L/2}\langle |(v(x)|\rangle \langle |v(y)|\rangle dxdy \ + 2\sigma^2 N\int_0^{L/2}\int_0^{L/2}\rho(x/L, y/L)dxdy$$

$$= 4\int_0^{L/2}\int_0^{L/2}\langle |(v(x)|\rangle \langle |v(y)|\rangle dxdy \ + 2\sigma^2 N L^2\int_0^{1/2}\int_0^{1/2}\rho(x', y')dx'dy'$$

Note the second double integral is a positive constant; the second term is a positive multiple of $\sigma^2 N L^2$ and the first term is $\langle d^*_{UD}\rangle^2$. The second moment becomes $\langle {d^*_{UD}}^2\rangle = \langle d^*_{UD}\rangle^2 + \rho_o\sigma^2 NL^2$, for some $\rho_o > 0$, and the variance is $\rangle d^*_{UD}\langle = \rho_o\sigma^2 NL^2 = \rho_o\sigma^2 l^2 N^3$. Since $\left\langle d^*_{UD}\right\rangle \rightarrow \sqrt{\frac{4}{9\pi}}\sigma\, lN^{3/2}$, the coefficient of variation is independent of σ, l and N, and it tends to a positive constant as stated. ∎

3. More general results (Daganzo and Smilowitz, 2000)

The result, $\langle p^* \rangle = \langle d^* \rangle / N = O(N^{1/2})$ when one holds δ constant, also holds for linear cost problems defined on "scalable graphs", but it does not extend to planar problems or problems in higher dimensions. The dependence on *N* is damped in higher dimensions. In two dimensions, the dependence is only of order *log*(*N*). In three dimensions $\langle p^* \rangle$ is bounded and the standard deviation of the optimum distance per point declines with *N*. This implies that in 2-D the optimum distance converges in probability to the expected value.

It is also possible to show that the asymptotic formulae of Sec. 3 depend on region shape, even if region size is held constant. That is, if one considers a region consisting of two intervals of equal length (*L*/2), separated by a much greater distance, *D*, with the depot in the middle then $\langle p^*_{UD} \rangle$ depends on *D* asymptotically. To see that this is true, note that in the optimal solution the total distance traveled by the depot flows increases linearly with $DN^{1/2}$, since the depot flow is proportional to $N^{1/2}$. This quantity cannot be neglected because it was shown at the end of Sec 3 that the internal distances within each sub-zone are $O(N^{1/2})$ when region size is held constant. Quite fortunately for practical applications, it turns out that shape does not have an asymptotic effect in 2-D. Thus, the 2-D TLP is quite similar to the TSP in that the optimum distance per point is shape-independent and appears to converge in probability to the asymptotic mean. Although the 2-D TLP distance per point is unbounded when one holds density constant (unlike in the case of the TSP), said distance increases with *N* so slowly that it may be treated as a constant for problems where *N* only varies by a factor of 10.

4. Acknowledgements

Research supported in part by the University of California Transportation Center.

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