

Some Properties of Decentralized Supply Chains

Yanfeng Ouyang and Carlos Daganzo
Institute of Transportation Studies and
Department of Civil and Environmental Engineering
University of California
Berkeley, CA 94720

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Abstract

This paper analyzes the bullwhip effect in decentralized, linear and time-invariant (LTI) supply chains. It generalizes existing results by broadening the class of policies and customer demand processes under consideration. The supply chain is modeled as a single-input, single-output control system driven by arbitrary demands. The paper discusses the appropriateness of various metrics for the bullwhip effect, and derives analytical conditions to predict its presence independently of the demand process. The paper also gives a formula for the variance of the order stream at any stage when the demand process is known and ergodic. Advance demand information (ADI) is shown to mitigate the bullwhip effect for general ordering policies.

In the supply chain literature, the term “bullwhip effect” refers to a phenomenon where the fluctuations in order sequence are usually greater upstream than downstream of the chain. Figure 1 shows an empirical example where the orders placed by a supplier are more variable than the actual quantities sold. In multi-echelon chains, even very steady customer demand can generate wildly fluctuating supplier orders several stages upstream. The upstream suppliers feel as if they were at the end of a bullwhip, where small perturbations at the handle (customers) cause huge movements at the tip (upstream suppliers). The phenomenon is also evident in macroeconomic data [18, 2, 19, 26, 25].

The bullwhip effect is of much practical importance. The term was originally coined by the Procter & Gamble Corporation to describe their empirical observations. It has also been described

by Callioni¹ as the “No.1 issue” faced by Supply Chain Services at Hewlett-Packard [4]. In business schools, “beer games” are widely used to demonstrate its existence and pernicious effects [29, 16, 20].

The bullwhip effect is important because it results in huge operating costs for upstream suppliers. These suppliers have to forecast demand, plan production and storage capacity, and control inventory based on the orders they receive. But these activities become inefficient with high order variability. With the bullwhip effect, a manufacturer would have to either: (i) set a high production capacity to satisfy its peak demands, wasting capacity during non-peak periods; (ii) set capacity at a lower level (e.g., slightly above the average demand rate), and either incur shortage in peak periods or carry large inventories; or (iii) adjust the capacity over time, incurring set-up costs. All these options imply either operating inefficiencies (high costs) or lack of responsiveness (poor customer service and loss of customer goodwill). Empirically, the bullwhip effect is estimated to inflate supply chain operating costs by 12.5 – 25% [21, 22]. If the bullwhip effect is eliminated, the U.S. grocery industry alone could save on the order of 30 billion dollars each year [8, 22].

1 Previous Work

Not surprisingly, the body of research on the bullwhip effect is extensive. The bullwhip effect was first recognized in the 1950s [23, 14, 15, 24]. Later, simulations and games [29, 16] revealed that it arises persistently, even if the games are unstructured. These reports attributed the causes of the bullwhip effect to “players’ systematic irrational behavior”, or to “misperception of feedback”.

Lee et al. [21, 22] looked for more satisfying explanations. They identified several operational causes and quantified their impacts for a single-echelon chain with an AR(1) customer demand process. The bullwhip effect was analyzed parametrically by comparing the variances of the orders placed by the supplier and the customer demand. Similar efforts, e.g. [1, 5, 6, 28], were later made to study variants of the problem for specific families of stationary demand processes. All these studies provide useful but, unfortunately, inconclusive insights because of their focus on single-echelons and their specific demand assumptions.

More recently, [9, 10, 11] used harmonic analysis to obtain analytical results for multi-echelon

¹Former director of Strategic Planning and Modeling (SPaM) at Hewlett-Packard

chains with general non-stationary demand inputs. These references show that all operationally efficient (rational) inventory control policies trigger the bullwhip effect, *independently of the demand process*. This is the essential reason for its prevalence. The references also show that if advance demand information and order commitments are allowed, the bullwhip effect can be eliminated while retaining efficiency. A decentralized ordering policy that achieves this goal for multi-echelon chains is presented.

In related but independent work, [12, 13] used the “transfer function” concept of control theory to derive variance formulae for a generalized family of order-up-to policies and numerically illustrate the bullwhip effect. This analysis did not include analytical results.

This paper uses the transfer function approach to generalize the results of [9, 10, 11, 12, 13] to a broader set of policies. Section 2 below formulates the problem; section 3 discusses the appropriateness of different metrics to measure the bullwhip effect; section 4 presents new analytical tests; section 5 gives some numerical examples; and section 6 discusses the effects of advance demand information.

2 Formulation

2.1 Basic notation

Figure 2 depicts a supply chain with $I + 1$ suppliers. Physical shipments arrive at the beginning of every time period; suppliers inspect their inventories during the period; replenishment orders are then placed at the end of the period and received by the upstream neighbors immediately.

Let $t = 0, 1, 2, \dots$ index the time periods, and $i = 1, 2, \dots, I + 1$ index the suppliers starting from the downstream end ($i = 0$ for the final customer). Define $u_i(t)$ as the quantity supplier i orders at the end of period t , and $N_i(t) := N_i^0 + \sum_{k=0}^{t-1} u_i(k)$ as the cumulative number of items that supplier i has ordered by the middle of period t , with N_i^0 being an initial value. Figure 3 shows a plot of $N_i(t)$. Note that its jumps equal the order quantities, and obviously

$$N_i(t + 1) = N_i(t) + u_i(t), \quad i = 0, 1, \dots, t = 0, 1, \dots \quad (1)$$

For convenience, we assume that (1) is also true for $t = -\infty, \dots, -1$, and that $N_i^0 = \sum_{k=-\infty}^{-1} u_i(k)$. From now on t refers to a generic period between $-\infty$ and ∞ .

Suppose that the goods ordered by supplier i always arrive after a constant lead time l_i (e.g., when $l_i = 0$ the goods ordered at the end of period t arrive at the beginning of period $t + 1$). Then by the middle of period t , the cumulative number of items received by supplier i is $V_i(t) := N_i(t - l_i)$; see the shifted curve in Figure 3. We define the *inventory position* of supplier i in period t , $x_i(t)$, as the vertical separation between curves $N_i(t)$ and $N_{i-1}(t)$; i.e., the difference between the cumulative orders placed and received:

$$x_i(t) := N_i(t) - N_{i-1}(t), \quad i = 1, 2, \dots \quad (2)$$

Similarly, the *in-stock inventory*, $y_i(t)$, is the vertical separation between $V_i(t)$ and $N_{i-1}(t)$; or equivalently,

$$y_i(t) := N_i(t - l_i) - N_{i-1}(t), \quad i = 1, 2, \dots \quad (3)$$

2.2 The system dynamics

Equations (1) – (3) yield the following expressions for the inventories at $t + 1$ as a function of the inventories at t and the current (and some previous) order quantities:

$$x_i(t + 1) = x_i(t) + u_i(t) - u_{i-1}(t), \quad i = 1, 2, \dots, \quad (4)$$

and

$$y_i(t + 1) = y_i(t) + u_i(t - l_i) - u_{i-1}(t), \quad i = 1, 2, \dots \quad (5)$$

These equations define the system dynamics for the supply chain when complemented with the suppliers' reorder policies; i.e., the recipes for determining $u_i(t), u_{i-1}(t)$ from the information available to them.

We consider decentralized supply chains where information is not shared. Thus, we assume that every supplier determines its order quantities with the demand and inventory information it has

experienced, and nothing else. At time t , supplier i knows the inventory records x_i, y_i up to period t , and orders u_{i-1}, u_i up to period $t - 1$. Thus, order $u_i(t)$ is based on the following information set:

$$\{x_i(t), x_i(t-1), \dots, x_i(-\infty); y_i(t), y_i(t-1), \dots, y_i(-\infty); u_i(t-1), \\ u_i(t-2), \dots, u_i(-\infty); u_{i-1}(t-1), u_{i-1}(t-2), \dots, u_{i-1}(-\infty)\}.$$

It turns out that all the u_i 's in this set are redundant. To see this, note from (4) with $k - 1$ substituted for t that $u_i(k-1) = x_i(k) - x_i(k-1) + u_{i-1}(k-1)$. Clearly, every u_i in the information set can be calculated with this formula from x_i 's and u_{i-1} 's that are also in the set. Thus the following set is equivalent:

$$\mathcal{I}_i(t) := \{x_i(t), x_i(t-1), \dots, x_i(-\infty); y_i(t), y_i(t-1), \dots, y_i(-\infty); \\ u_{i-1}(t-1), u_{i-1}(t-2), \dots, u_{i-1}(-\infty)\}.$$

We shall consider a broad family of policies based on $\mathcal{I}_i(t)$. This set enlarges the one in [9, 10, 11], which consisted of the history of orders received, $u_{i-1}(t-1), u_{i-1}(t-2), \dots, u_{i-1}(-\infty)$, and the most recent inventory position, $x_i(t)$.

As in these references, we focus on ordering policies that are:

1. *Proper*; i.e., for any steady demand the supplier inventories tend to a nominal equilibrium independent of the initial conditions. Improper policies, which amplify perturbations over time, usually entail unbounded costs; they would not be appealing to a rational supplier.
2. *Linear and time-invariant (LTI)*; i.e., $u_i(t)$ is a time-independent linear function of the elements in $\mathcal{I}_i(t)$. Linear policies are important as they are often used in practice. Nonlinear policies can also be studied in linearized form to reveal how a supply chain responds to small perturbations from an equilibrium state.

To express LTI policies in a simple way, let P be the unit shift operator for a time series and P^k its k -fold application; i.e., $P^k x_i(t) = x_i(t-k), \forall t$ and $\forall k = 0, 1, \dots$. The most general LTI

expression based on $\mathcal{I}_i(t)$ is:

$$u_i(t) = \gamma_i + A_i(P)x_i(t) + B_i(P)y_i(t) + C_i(P)u_{i-1}(t-1), \quad i = 1, 2, \dots, \quad (6)$$

where γ_i is a real number; and $A_i(P), B_i(P)$ and $C_i(P)$ are polynomials with real coefficients: $A_i(P) = a_0^i + a_1^i P + a_2^i P^2 + \dots$, $B_i(P) = b_0^i + b_1^i P + b_2^i P^2 + \dots$, $C_i(P) = c_0^i + c_1^i P + c_2^i P^2 + \dots$. Polynomials $A_i(P)$ and $B_i(P)$ indicate the influence of inventory history on ordering decisions, and $C_i(P)$ the influence of orders received. For example, $A_i(P) = -1$, $B_i(P) = 0$, and $C_i(P) = l_i \frac{1}{r_i} (1 + P + \dots + P^{r_i-1})$ denote an order-up-to policy, where the “up-to level” is forecasted by a moving-average of orders received over r_i periods.

We note that iterations of (4) – (6) may lead to negative orders and negative inventories. Obviously, practical policies should avoid negativity, as well as the bullwhip effect. It is known, however, that policies that avoid the bullwhip effect do not seriously suffer from the negativity problem, and some do not have it at all [9, 10]. Numerical simulations had hinted at these facts [6]. Thus, we do not screen for negativity in the analysis that follows.

2.3 Steady states

Since we assumed that policies are proper, a nominal equilibrium state must exist in which all suppliers place orders of constant size u^∞ , while the inventory positions x_i^∞ and the in-stock inventories y_i^∞ are steady. We assume for convenience that the system is in this equilibrium for all $t \leq 0$.

We now look at the properties of the steady state. The steady-state variables $(u^\infty, x_i^\infty, y_i^\infty)$ should obviously satisfy the system dynamics. They trivially satisfy (4) and (5), and must also satisfy (6); i.e.,

$$u^\infty = \gamma_i + A_i(P)x_i^\infty + B_i(P)y_i^\infty + C_i(P)u^\infty, \quad i = 1, 2, \dots. \quad (7)$$

Since $(u^\infty, x_i^\infty, y_i^\infty)$ are time-invariant, they are not changed by time shifts. Hence if we replace

$P^k, \forall k$ by $P^0 \equiv 1$ in (7), the equality continues to hold. Thus, we have:

$$u^\infty = \gamma_i + A_i(1)x_i^\infty + B_i(1)y_i^\infty + C_i(1)u^\infty, i = 1, 2, \dots. \quad (8)$$

We also see from the geometry of Figure 3 that x_i^∞, y_i^∞ and u^∞ must satisfy Little's formula:

$$x_i^\infty - y_i^\infty = l_i u^\infty.$$

This, together with (8), implies that

$$\begin{aligned} [1 - A_i(1)l_i - C_i(1)]u^\infty &= \gamma_i + [A_i(1) + B_i(1)]y_i^\infty, i = 1, 2, \dots, \\ [1 + B_i(1)l_i - C_i(1)]u^\infty &= \gamma_i + [A_i(1) + B_i(1)]x_i^\infty, i = 1, 2, \dots; \end{aligned} \quad (9)$$

i.e., there is a univocal relationship between each of the two inventories and the order quantity at equilibrium.

Note that for a policy to be proper, A_i and B_i must satisfy $A_i(1) + B_i(1) \neq 0$; otherwise (9) does not uniquely define x_i^∞ and y_i^∞ for every u^∞ . Thus, when a policy is proper, (9) can be inverted and the equilibrium inventories can be expressed as a function of the order size:

$$\begin{aligned} y_i^\infty &= \frac{-\gamma_i}{A_i(1) + B_i(1)} + \frac{1 - A_i(1)l_i - C_i(1)}{A_i(1) + B_i(1)}u^\infty, i = 1, 2, \dots, \\ x_i^\infty &= \frac{-\gamma_i}{A_i(1) + B_i(1)} + \frac{1 + B_i(1)l_i - C_i(1)}{A_i(1) + B_i(1)}u^\infty, i = 1, 2, \dots. \end{aligned} \quad (10)$$

Equations (10) give the suppliers' preferred in-stock inventory and inventory positions when they receive and place steady orders of size u^∞ . The coefficient of u^∞ in the second function,

$$\frac{d(x_i^\infty)}{d(u_i^\infty)} = \frac{1 + B_i(1)l_i - C_i(1)}{A_i(1) + B_i(1)}, \quad (11)$$

is the "inventory gain" of [9] — i.e., "the marginal change in the equilibrium inventory position for a unit change in the equilibrium demand". The intercept terms of (10) are the inventories kept

with zero demand:

$$x_i^\infty = y_i^\infty = \frac{-\gamma_i}{A_i(1) + B_i(1)}, i = 1, 2, \dots .$$

2.4 Deviations from equilibrium

To facilitate analysis, we express the system dynamics in terms of deviations (errors) from an equilibrium. (This is standard practice especially for nonlinear policies.) To this end, let $\bar{x}_i(t) := x_i(t) - x_i^\infty$, $\bar{y}_i(t) := y_i(t) - y_i^\infty$, and $\bar{u}_i(t) := u_i(t) - u^\infty$. Note that $\bar{x}_i(t) = \bar{y}_i(t) = \bar{u}_i(t) = 0$ for all $t = -\infty, \dots, 0$ since the system is assumed to start from equilibrium.

The usual manipulations (e.g., subtracting (7) from (6), etc.) yield:

$$\bar{u}_i(t) = A_i(P)\bar{x}_i(t) + B_i(P)\bar{y}_i(t) + C_i(P)\bar{u}_{i-1}(t-1), i = 1, 2, \dots , \quad (12)$$

$$\bar{x}_i(t+1) = \bar{x}_i(t) + \bar{u}_i(t) - \bar{u}_{i-1}(t), i = 0, 1, \dots , \quad (13)$$

$$\bar{y}_i(t+1) = \bar{y}_i(t) + \bar{u}_i(t-l_i) - \bar{u}_{i-1}(t), i = 0, 1, \dots . \quad (14)$$

Note the similarity between (12) – (14) and (4) – (6), except for the absence of the intercept term. The system dynamics equations are now homogeneous and ready for analysis.

3 Metrics for the Bullwhip Effect

3.1 A preferred metric

Note from (12) – (14) that any realization of customer demand $\{\bar{u}_0(t)\}_{t=0}^\infty$ defines a unique upstream order sequence $\{\bar{u}_I(t)\}_{t=0}^\infty$. Thus, an easily understood bullwhip effect metric is the ratio of the root mean square errors (RMSE) of (i) the order sequence received by the most upstream supplier, $\{\bar{u}_I(t)\}_{t=0}^\infty$, and (ii) the customer demand, $\{\bar{u}_0(t)\}_{t=0}^\infty$. Since we have neither full knowledge nor control over the realization of customer demand, we shall use the *worst-case RMSE amplification factor*, W_I , across all possible customer demand sequences:

Definition 1. Supplier $I+1$ ($I > 0$) in a supply chain described by (12) – (14) is said to experience

no bullwhip effect if

$$W_I := \sup_{\forall \{\bar{u}_0(t)\} \neq 0} \left[\frac{(\sum_{t=0}^{\infty} \bar{u}_I^2(t))^{\frac{1}{2}}}{(\sum_{t=0}^{\infty} \bar{u}_0^2(t))^{\frac{1}{2}}} \right] \leq 1. \quad (15)$$

The quantity W_I is also called the L_2 gain or root mean square gain in the control literature.² By taking the worst-case amplification across every possible input sequence $\{\bar{u}_0(t)\}_{t=0}^{\infty}$, we ensure that W_I is an intrinsic property of the supply chain and does not depend on the customer demand process.

Any realization of $\{\bar{u}_0(t)\}_{t=0}^{\infty}$ can be decomposed by a discrete Fourier transform into a set of pure harmonic components, $\mathcal{A}_0(w)e^{-jwt}$ (where $j = \sqrt{-1}$), each with an angular frequency $w \in [0, 2\pi)$ and a complex amplitude $\mathcal{A}_0(w) \in \mathcal{C}$. Because (12) – (14) are linear and time-invariant, the following two things hold: (i) For any harmonic component, the output from each supplier stage of the linear chain is also harmonic with the same frequency but a different amplitude, $\mathcal{A}_i(w)e^{-jwt}$, $i = 1, 2, \dots, I$; (ii) the combined output, $\{\bar{u}_I(t)\}_{t=0}^{\infty}$, is the superposition of the harmonic outputs at the final stage, $\mathcal{A}_I(w)e^{-jwt}$.

We define a “transfer function”, $T_I(\cdot)$, such that

$$T_I(e^{jw}) = \frac{\mathcal{A}_I(w)}{\mathcal{A}_0(w)} = \frac{\mathcal{A}_I(w)}{\mathcal{A}_{I-1}(w)} \cdots \frac{\mathcal{A}_i(w)}{\mathcal{A}_{i-1}(w)} \cdots \frac{\mathcal{A}_1(w)}{\mathcal{A}_0(w)}.$$

If we additionally define the “stage- i transfer function” $T_{i-1,i}(\cdot)$, by $T_{i-1,i}(e^{jw}) = \mathcal{A}_i(w)/\mathcal{A}_{i-1}(w)$, then

$$T_I(e^{jw}) := \prod_{i=1}^I T_{i-1,i}(e^{jw}). \quad (16)$$

Section 4 presents a standard procedure to derive the transfer functions.

The modulus of the transfer function $|T_I(e^{jw})|$ is the amplification factor for harmonic component w . Its maximum across all $w \in [0, 2\pi)$ (the H_{∞} norm in the frequency domain) is obviously the worst-case amplification across the superposition of all possible inputs. Thus,

$$W_I = \sup_{\forall w \in [0, 2\pi)} |T_I(e^{jw})|. \quad (17)$$

²This is unrelated to the inventory gain defined earlier.

The equivalence of (15) and (17) is well known.

3.2 Homogeneous chains

In homogeneous chains, all suppliers are alike. Thus, the behavior of the chain can be inferred from the behavior of one stage. Let the worst-case RMSE amplification for one stage be:

$$W_1 := \sup_{\forall w \in [0, 2\pi)} |T_{0,1}(e^{jw})|.$$

Then we have:

Theorem 1. *If a chain is homogeneous and $W_1 \leq 1$, then $W_I \leq 1$ for all I .*

Proof. For homogeneous chains, (16) reduces to $|T_I(e^{jw})| = |T_{0,1}(e^{jw})|^I$. Since $\sup_w \{|T_{0,1}(e^{jw})|^I\} = \{\sup_w |T_{0,1}(e^{jw})|\}^I$, it follows that $W_I = (W_1)^I$. \square

This shows that if a policy avoids the bullwhip effect for one stage, it avoids it for many.

3.3 Problems with other metrics

Theorem 1 does not hold if the bullwhip effect is defined by any other metric; e.g. those proposed in the literature [22, 5, 6, 28, 12, 13]. This is important because most of these works study single-echelon chains (i.e., $I = 1$). The conclusions drawn with these metrics cannot be generalized to the multi-echelon case.

3.3.1 Other metrics

One school of research [22, 5, 6, 28] studies single-echelon chains in the time domain with an “order variance amplification” metric. These works always assume specific demand processes (e.g., stationary AR(1)) and then obtain closed-form formulae. Unfortunately, results for cases where the demand process is not known a priori have not yet been developed. Furthermore, multi-echelon work by this school is sparse and even more specific. Only [5] reports some results, but does so for a very specific policy and demand process with the introduction of lower bounds.

Other researchers [12, 13] worked in the frequency domain with the H_2 norm. (This is the un-weighted average of squared amplitude amplification across the frequency spectrum.) Unfortunately, as the example in Section 3.3.2 shows, this metric will sometimes predict no bullwhip effect when the variances are really being amplified.

3.3.2 Example

Suppose the suppliers in a homogeneous chain use the “general replenishment rule”³ in [12], and the customer demand exhibits both seasonal and short-term variations. The customer demand error has two sinusoidal components: one with amplitude 1.0 and angular frequency 0.05π , and another with amplitude 2.0 and angular frequency 0.88π ; see Figure 4(a).

A simple simulation with this demand yields the results shown in Figures 4(b)–(d). Note how the variance of the order sequences decreases for a few suppliers, and then increases. The reason is that the policy dampens the initially larger short-term fluctuations but amplifies the initially smaller seasonal fluctuations. This example clearly illustrates that variance amplification predictions obtained for single-echelon chains, e.g. as in [22, 5, 6, 28], cannot be extrapolated to multi-echelon chains. It also illustrates the inadequacy of the H_2 norm, since the general replenishment rule in [12] amplifies variances, even though its H_2 norm is less than 1.

The example is not contrived. The problem arises with any customer demand process whose spectrum is heavily weighted toward the frequencies that are damped but includes at least one frequency w with $|T_I(e^{jw})| > 1$. Clearly, in multi-echelon chains, the *only* metric that correctly diagnoses the bullwhip effect for any demand process is the worst-case RMSE amplification W_I . This metric has other advantages. For example, the single-stage RSME amplification factor arising from any stochastic process with a full spectrum⁴ must converge to W_1 as I increases, *for all realizations*. We now present formulae for $T_I(\cdot)$ and W_I , and a simple sufficient condition for existence of the bullwhip effect .

³For more details see [12], page 584.

⁴Processes without a full spectrum are idealizations that do not arise in practice.

4 Formulae and Tests

As is conventional, we derive the transfer function $T_I(\cdot)$ by applying the z-transform⁵ to (12) – (14). Denote the z-transforms of the error sequences by $X_i(z) := Z\{\bar{x}_i(t)\}$, $Y_i(z) := Z\{\bar{y}_i(t)\}$, and $U_i(z) := Z\{\bar{u}_i(t)\}$. Recall that the z-transform is a linear operator, and $Z\{P^k \bar{x}_i(t)\} = Z\{\bar{x}_i(t-k)\} = z^{-k} X_i(z)$. Therefore $Z\{A_i(P)\bar{x}_i(t)\} = A_i(z^{-1})X_i(z)$ and similarly $Z\{B_i(P)\bar{y}_i(t)\} = B_i(z^{-1})Y_i(z)$, $Z\{C_i(P)\bar{u}_{i-1}(t-1)\} = C_i(z^{-1})z^{-1}U_{i-1}(z)$. Thus, if we now apply the z-transform to both sides of (12) – (14), we obtain:

$$\begin{aligned} U_i(z) &= A_i(z^{-1})X_i(z) + B_i(z^{-1})Y_i(z) + \frac{C_i(z^{-1})}{z}U_{i-1}(z), \forall i, \\ (z-1)X_i(z) &= U_i(z) - U_{i-1}(z) + z\bar{x}_i(0), \forall i, \\ (z-1)Y_i(z) &= z^{-l_i}U_i(z) - U_{i-1}(z) + z\bar{y}_i(0), \forall i. \end{aligned}$$

We have assumed that the system starts from the equilibrium state at $t = 0$, and thus $\bar{x}_i(0) = \bar{y}_i(0) = \bar{u}_i(0) = 0$. Simple manipulations of these equations reveal that:

$$U_i(z) = T_{i-1,i}(z)U_{i-1}(z), i = 1, 2, \dots, \quad (18)$$

where

$$T_{i-1,i}(z) = \frac{z^{-1}C_i(z^{-1}) - (z-1)^{-1}[A_i(z^{-1}) + B_i(z^{-1})]}{1 - (z-1)^{-1}[A_i(z^{-1}) + z^{-l_i}B_i(z^{-1})]}. \quad (19)$$

Equation (18) shows how supplier i transforms its input into an output. By assumption, the policy of supplier i is stable in time (it is proper). This happens if all the poles of $T_{i-1,i}(z)$ are within the unit circle of the complex plane, $\{z : |z| < 1, z \in \mathcal{C}\}$.

Equations (18) and (19) also yield a formula for the variance of the orders placed by any supplier in chains with ergodic demand. Since variance equals mean square error for an ergodic sequence, and since the mean square error of $U_i(z)$ is $\frac{1}{2\pi} \int_{-\pi}^{\pi} U_i(e^{jw}) \cdot U_i(e^{-jw}) dw$, we have

⁵The z-transform of a given discrete sequence $\{f(t)\}_{-\infty}^{\infty}$ is given by $Z\{f(t)\} := \sum_{-\infty}^{\infty} f(t)z^{-t}$. It is essentially the discrete Fourier transform after the substitution $z = e^{jw}$.

Theorem 2. *If the customer demand is ergodic, the variance of orders placed by supplier I is*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} U_I(e^{jw}) \cdot U_I(e^{-jw}) dw. \quad (20)$$

This formula depends on the character of the input process, U_0 , as can be seen from (18) and (19).

When the input demand is not known, we can still test for the bullwhip effect. As indicated by Definition 1, (16) and (17), the bullwhip effect arises if and only if

$$\sup_{w \in [0, 2\pi)} |T_I(e^{jw})| = \sup_{w \in [0, 2\pi)} \prod_{i=1}^I |T_{i-1,i}(e^{jw})| > 1. \quad (21)$$

This test is rigorous but tedious and qualitatively obscure. Therefore, simpler but still demand-independent tests have been sought.

It was shown in [9, 10, 11] from the perspective of conservation laws that any homogeneous chain with positive inventory gain must experience the bullwhip effect. We conclude this section with a generalization of this statement.

Theorem 3. *Supplier $I+1$ in an LTI supply chain described by (12) – (14) experiences the bullwhip effect if*

$$\sum_{i=1}^I \frac{1 + B_i(1)l_i - C_i(1)}{A_i(1) + B_i(1)} > 0. \quad (22)$$

Proof. The proof is similar to the proof for Theorem 4.1 in [7] and Theorem 3.1 in [27]. Let $z = e^\sigma, \sigma \in \mathcal{C}$, then $Re(\sigma) > 0 \Leftrightarrow |z| > 1$. The assumption on properness and time stability of the system implies that any σ that is on the right half of the complex plane cannot be a pole of $T_I(e^\sigma)$. Thus, $|T_I(e^\sigma)|$ is bounded from above in $\{\sigma : Re(\sigma) > 0, \sigma \in \mathcal{C}\}$. Boyd and Desoer [3] showed that with such boundedness property $\log |T_I(e^\sigma)|$ is subharmonic with regard to σ and satisfies the Poisson Inequality:

$$\log |T_I(e^s)| \leq \frac{1}{\pi} \int_{-\infty}^{+\infty} \log |T_I(e^{jw})| \frac{sdw}{s^2 + w^2}, \forall s \in (0, \infty). \quad (23)$$

Divide both sides by the positive real number s , and let $s \rightarrow 0^+$, (23) gives the inequality below:

$$\begin{aligned} \lim_{s \rightarrow 0^+} \frac{1}{s} \log |T_I(e^s)| &\leq \lim_{s \rightarrow 0^+} \frac{1}{\pi} \int_{-\infty}^{+\infty} \log |T_I(e^{jw})| \frac{dw}{s^2 + w^2} \\ &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \log |T_I(e^{jw})| \frac{dw}{w^2} \end{aligned} \quad (24)$$

Note that $|T_I(e^s)| = \prod_{i=1}^I |T_{i-1,i}(e^s)|$; therefore

$$\log |T_I(e^s)| = \sum_{i=1}^I \log |T_{i-1,i}(e^s)|,$$

and

$$\lim_{s \rightarrow 0^+} \frac{1}{s} \log |T_I(e^s)| = \sum_{i=1}^I \lim_{s \rightarrow 0^+} \frac{1}{s} \log |T_{i-1,i}(e^s)|. \quad (25)$$

It is easy to verify that for $i = 1, 2, \dots, I$,

$$T_{i-1,i}(e^s)|_{s=0} = 1, \quad [T_{i-1,i}(e^s)]'_s|_{s=0} = \frac{1 + B_i(1)l_i - C_i(1)}{A_i(1) + B_i(1)}.$$

By Taylor expansion in the neighborhood of $s = 0^+$,

$$\begin{aligned} T_{i-1,i}(e^s) &= T_{i-1,i}(e^s)|_{s=0} + [T_{i-1,i}(e^s)]'_s|_{s=0} \cdot s + o(s) \\ &= 1 + \frac{1 + B_i(1)l_i - C_i(1)}{A_i(1) + B_i(1)} \cdot s + o(s). \end{aligned}$$

As $s \rightarrow 0^+$, $\left| \frac{1+B_i(1)l_i-C_i(1)}{A_i(1)+B_i(1)} \cdot s + o(s) \right| \ll 1$, therefore

$$\begin{aligned} |T_{i-1,i}(e^s)| &= \left| 1 + \frac{1 + B_i(1)l_i - C_i(1)}{A_i(1) + B_i(1)} \cdot s + o(s) \right| \\ &= 1 + \frac{1 + B_i(1)l_i - C_i(1)}{A_i(1) + B_i(1)} \cdot s + o(s). \end{aligned}$$

By l'Hôpital's Rule,

$$\begin{aligned}
\lim_{s \rightarrow 0^+} \frac{1}{s} \log |T_{i-1,i}(e^s)| &= \lim_{s \rightarrow 0^+} \frac{|T_{i-1,i}(e^s)|'_s}{|T_{i-1,i}(e^s)|} \\
&= \lim_{s \rightarrow 0^+} \frac{\frac{1+B_i(1)l_i-C_i(1)}{A_i(1)+B_i(1)} + O(s)}{1 + \frac{1+B_i(1)l_i-C_i(1)}{A_i(1)+B_i(1)} \cdot s + o(s)} \\
&= \frac{1 + B_i(1)l_i - C_i(1)}{A_i(1) + B_i(1)}. \tag{26}
\end{aligned}$$

Substituting (26) into (25) then (24), we have

$$\sum_{i=1}^I \frac{1 + B_i(1)l_i - C_i(1)}{A_i(1) + B_i(1)} \leq \frac{1}{\pi} \int_{-\infty}^{+\infty} \log |T_I(e^{jw})| \frac{dw}{w^2}. \tag{27}$$

When $\sum_{i=1}^I \frac{1+B_i(1)l_i-C_i(1)}{A_i(1)+B_i(1)} > 0$, (27) yields $\int_{-\infty}^{+\infty} \log |T_I(e^{jw})| \frac{dw}{w^2} > 0$. There must exist a frequency, $\bar{w} \in (-\infty, \infty)$, that satisfies $\log |T_I(e^{j\bar{w}})| > 0$ (or equivalently $|T_I(e^{j\bar{w}})| > 1$). Since $|T_I(e^{jw})|$ as a function of w has a period of 2π , there must exist a frequency $w^* \in [0, 2\pi)$ such that $|T_I(e^{jw^*})| > 1$. This completes the proof. \square

The following corollary is the result in [9, 10].

Corollary 1. *When the supply chain is homogeneous, the bullwhip effect exists if*

$$\frac{1 + B(1)l - C(1)}{A(1) + B(1)} > 0. \tag{28}$$

5 Numerical Examples

This section presents numerical results for three types of policies: (i) “order-up-to” with the “level” adjusted based on a 2-period moving-average of orders received (as in Section 2); (ii) “Kanban” with in-stock inventories (like the “general rule” in [12]); and (iii) “order-based” (like some of the rules in [9]). For simplicity we only consider multi-echelon homogeneous chains with lead time $l = 2$ at every stage.

5.1 Policies

5.1.1 Order-up-to

We consider the following special case (with $r_i = 2$) of the policy examined at the end of Section 2.2. In terms of errors, the policy is:

$$\bar{u}_i(t) = -\bar{x}_i(t) + \bar{u}_{i-1}(t-1) + \bar{u}_{i-1}(t-2), \forall i, t.$$

Its polynomials are: $A(P) = -1$, $B(P) = 0$, and $C(P) = 1 + P$. Thus, (28) reduces to:

$$\frac{1 + B(1)l - C(1)}{A(1) + B(1)} = \frac{1 + 0 - 2}{-1 + 0} = 1 > 0,$$

and we see immediately that the bullwhip effect exists.

Since the chain is homogeneous, we can also apply Theorem 1. Thus, W_1 determines the bullwhip effect. To find W_1 , we write from (19) the transfer function for one stage:

$$T_{0,1}(z) = \frac{2z^2 - 1}{z^3}, \quad (29)$$

Obviously, the policy is proper since all its poles are 0. Then, we can plot $|T_{0,1}(e^{jw})|$ over $w \in [0, 2\pi)$ to find its maximum; see Figure 5(a). The result is $W_1 = 3.0 > 1$, confirming that the bullwhip effect arises.

5.1.2 Kanban

We consider here a case where orders are partly based on in-stock inventory:

$$\bar{u}_i(t) = -\bar{x}_i(t)/8 - \bar{y}_i(t)/8 + \bar{u}_{i-1}(t-1)/2 + \bar{u}_{i-1}(t-2)/2, \forall i, t.$$

Now, $A(P) = -1/8$, $B(P) = -1/8$, and $C(P) = (1 + P)/2$, and the inventory gain (28) is

$$\frac{1 + B(1)l - C(1)}{A(1) + B(1)} = \frac{1 - 1/4 - 1}{-1/8 - 1/8} = 1 > 0.$$

Therefore, the bullwhip effect exists.

The transfer function for one stage (19) reduces to

$$T_{0,1}(z) = \frac{6z^2 - 4}{8z^3 - 7z^2 + 1},$$

which is again proper. A plot of $|T_{0,1}(e^{jw})|$ over $w \in [0, 2\pi)$ is shown in Figure 5(b). Note that $W_1 = 1.68 > 1$, as expected.

5.1.3 Order-based

A simple policy of this type is:

$$\bar{u}_i(t) = 0.5\bar{u}_i(t-1) + 0.3\bar{u}_{i-1}(t-1) + 0.2\bar{u}_{i-1}(t-2), \forall i, t.$$

The polynomials of this policy are $A(P) = -1/2$, $B(P) = 0$ and $C(P) = -1/5$. The inventory gain satisfies:

$$\frac{1 + B(1)l - C(1)}{A(1) + B(1)} = \frac{1 + 0 + 1/5}{-1/2 + 0} = -12/5 < 0.$$

Therefore, we cannot judge from this inequality whether the bullwhip effect arises.

The definitive answer is found from Theorem 1. The transfer function for one stage is now:

$$T_{0,1}(z) = \frac{3z + 2}{10z^2 - 5z}.$$

It is proper, and the plot of $|T_{0,1}(e^{jw})|$ over $w \in [0, 2\pi)$ is shown in Figure 5(c). Note that $W_1 = 1.00$. Thus, the bullwhip effect does not arise with this policy. The bullwhip effect does not arise for *any* realization of demand, and for chains with *any* number of stages.

If the customer demand, i.e. $U_0(z)$, is known in any of these examples, we can also find $U_i(z)$ with (18) and (19). Then we can apply Theorem 2, and obtain an exact expression for the variance of the order stream at any stage. For the example of Section 5.1.1, if the customer demand follows

a standard i.i.d. Gaussian process, i.e. where $U_0(z) = 1, \forall |z| = 1$, then $U_i(z) = (\frac{2z^2-1}{z^3})^i$, and

$$\begin{aligned}\text{Var}(\bar{u}_i) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} U_i(e^{jw}) \cdot U_i(e^{-jw}) dw \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [5 - 4 \cos(2w)]^i dw.\end{aligned}$$

The results of this formula for $i = 1$ to 5 are depicted by the square markers in Figure 6. Similar formulae can be developed for all our policies.

5.2 Simulations

Simulations are now used to illustrate the bullwhip effect for these policies. We consider the following four demand processes:

1. a family of three stationary AR(1) processes:

$$u_0(t+1) = \rho \cdot u_0(t) + (1 - \rho) \cdot \varepsilon(t), \forall t,$$

for $\rho = 0, 0.4, 0.8$, with i.i.d. Gaussian error terms $\varepsilon(t)$ of mean 0 and variance 1.

2. a time-dependent process obtained by superposing our AR(1) process with $\rho = 0.4$ and a sinusoidal wave:

$$u_0(t+1) = 0.4 \cdot u_0(t) + 0.6 \cdot \varepsilon(t) + 7 \cdot \sin(0.95\pi t), \forall t.$$

Customer order sequences were generated from these demand processes, and the resulting supplier orders simulated with (12) – (14). These simulations match the prediction of (20) to within the statistical tolerance of the simulation as expected — the solid line in Figure 6 represents simulated values. Figure 7 gives the exact/simulated amplification factors for the RMSE at each supplier stage. Each diagram corresponds to a policy, and each curve to a demand process.

We observe the following:

1. In all cases, the RMSE amplification factor at stage i converges from below to W_1 as $i \rightarrow \infty$.

This is expected, as pointed out earlier, because W_1 is the worst-case amplification and bounds from above the result for all possible inputs. As discussed in Section 3, the frequency component corresponding to W_1 becomes dominant.

2. For every policy, the curves are far apart for the first several stages ($i \leq 5$). But, as i increases, the curves converge. Thus, customer demand influences the RMSE amplification factor, but mostly in the stages just upstream of the customer. Farther upstream, where the bullwhip effect is most significant, the critical contributing factor is the policy.
3. Reference [5] (Theorem 3.2) provided a variance amplification lower bound for multi-echelon chains and a class of order-up-to policies with i.i.d. customer demands. This result only applies to our first policy, with $\rho = 0$. It is expressed in RMSE terms as the dotted line in Figure 6. The bound is also plotted on Figure 7(a), which also includes curves for processes with correlated orders.

6 Advance Demand Information

Theorem 3 (Section 4) established a relationship between the bullwhip effect and positive inventory gain. To eliminate the former, we must restrict the latter. It is known, however, that this restriction on inventory gain does not have to be as severe for policies with advance demand information (ADI) [9]. Indeed, [9] proposed a family of ADI policies that avoid the bullwhip effect with any desired inventory gain. This section generalizes these results. It examines the effect of ADI on, both, inventory gain and the bullwhip effect, for general policies. ADI has also been shown to have the same beneficial effect as a reduction in lead time [17].

6.1 Formulation

To provide advance demand information (ADI), suppliers inform their immediate upstream neighbors the orders they will place in some future periods; this information is then integrated into the ordering policies.

Consider a generic stage i . With $h_i = 1, 2, \dots$ periods of ADI (i.e., $u_{i-1}(t), \dots, u_{i-1}(t+h_i-1)$), supplier i places order $u_i(t)$ based on the following set of available information:

$$\begin{aligned} \mathcal{I}_i(t) := & \{x_i(t), x_i(t-1), \dots, x_i(-\infty); y_i(t), y_i(t-1), \dots, y_i(-\infty); \\ & u_{i-1}(t+h_i-1), \dots, u_{i-1}(t), \dots, u_{i-1}(-\infty)\}, \end{aligned}$$

This information is marked as the thick dashed line in Figure 8(a). The most general policy based on this information is:

$$u_i(t) = \gamma_i + A_i(P)x_i(t) + B_i(P)y_i(t) + C_i(P)u_{i-1}(t+h_i-1), \quad \forall i, t. \quad (30)$$

Note this is the same as (6) except that committed future orders have been incorporated into the policy.

It is easy to see that the steady states of (6) and (30) are the same. Thus, the inventory gain of this policy is still given by (11). It is also shown in the appendix that the single-stage transfer function formula (19) now becomes

$$\tilde{T}_{i-1,i}(z, h_i) = \frac{z^{-1}C_i(z^{-1}) - z^{-h_i}(z-1)^{-1}[A_i(z^{-1}) + B_i(z^{-1})]}{1 - (z-1)^{-1}[A_i(z^{-1}) + z^{-h_i}B_i(z^{-1})]}.$$

Note that the introduction of ADI does not change the poles of the transfer function; it preserves “properness”.

Consider the entire chain, its transfer function is now:

$$\tilde{T}_I(z, \mathbf{h}) := \prod_{i=1}^I \tilde{T}_{i-1,i}(z, h_i),$$

where $\mathbf{h} := \{h_1, h_2, \dots, h_I\}$; the L_2 gain is:

$$\tilde{W}_I(\mathbf{h}) := \sup_{w \in [0, 2\pi)} |\tilde{T}_I(e^{jw}, \mathbf{h})|. \quad (31)$$

Theorem 3 now becomes:

Theorem 4. *Supplier $I + 1$ in an LTI supply chain with ADI policy (30) experiences the bullwhip effect if*

$$\sum_{i=1}^I \frac{1 + B_i(1)l_i - C_i(1)}{A_i(1) + B_i(1)} > \sum_{i=1}^I h_i. \quad (32)$$

Proof. See appendix. □

Note that the first term is the summation of inventory gains at each supplier stage. Thus, h_i periods of ADI committed at stage $i - 1$ ($\forall i = 1, 2, \dots, I$) increases the instability barrier by $\sum_{i=1}^I h_i$. The advantage of ADI is that it raises this barrier, while allowing suppliers to operate with the same inventory gain as before.

6.2 Examples

We investigate the two policies of Section 5 that exhibit the bullwhip effect. We show that ADI eliminates the bullwhip effect in both cases.

For the order-up-to policy, $A(P) = -1$, $B(P) = 0$, and $C(P) = 1 + P$; from (31) the one-stage transfer function becomes

$$\tilde{T}_{0,1}(z, h) = \frac{z^2 + z^{2-h} - 1}{z^3}.$$

From (31), we find that $\tilde{W}_1(2) = 1.0$, which now satisfies Theorem 1. Thus, introducing ADI with $h = 2$ eliminates the bullwhip effect.

For the Kanban policy, $A(P) = -1/8$, $B(P) = -1/8$, and $C(P) = (1 + P)/2$. The one-stage transfer function is

$$\tilde{T}_{0,1}(z, h) = \frac{4z^2 + 2z^{2-h} - 4}{8z^3 - 7z^2 + 1}.$$

From (31), we find that $\tilde{W}_1(3) = 1.0$ which, again, satisfies Theorem 1. Thus, in this case too, ADI (with $h = 3$) eliminates the bullwhip effect.

These examples suggest but do not prove that ADI can eliminate the bullwhip effect for any decentralized LTI policy.

7 Conclusions and Future Research

This paper has presented a system control framework for analyzing the bullwhip effect in decentralized, linear, time-invariant, and multi-echelon supply chains. The formulation can be easily extended to general supply networks. By modeling the supply chain as a single-input, single-output system, we proposed appropriate metrics for the bullwhip effect, and derived analytical conditions that predict its presence. These conditions generalize previous findings. We also showed the beneficial effect of advance demand information for general ordering policies.

A natural extension of this work would relax its deterministic, linear, and time-invariant assumptions. It would allow for stochastic system operations such as transportation losses, random lead times, and policy alternations. Our findings also open the door for the development of decentralized contracting schemes that could drive an entire supply network toward optimality.

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Appendix

We first express policy (30) into a form similar to (6) by redefining the inventory and order variables. First, define $D_i(P) := 1 + P + \dots + P^{h_i-1}$ so that the total advance orders committed by supplier $i - 1$ at time t equals $D(P)u_{i-1}(t + h_i - 1) = u_{i-1}(t + h_i - 1) + \dots + u_{i-1}(t)$. Then, define the following variables:

$$\begin{aligned} u'_i(t) &:= u_i(t), \forall t, \\ u'_{i-1}(t-1) &:= u_{i-1}(t + h_i - 1), \forall t, \\ x'_i(t) &:= x_i(t) - D(P)u_{i-1}(t + h_i - 1) = x_i(t) - D(P)u'_{i-1}(t-1), \forall t, \\ y'_i(t) &:= y_i(t) - D(P)u_{i-1}(t + h_i - 1) = y_i(t) - D(P)u'_{i-1}(t-1), \forall t. \end{aligned}$$

By substituting these new variables in (30), we find:

$$\begin{aligned} u'_i(t) &= \gamma_i + A_i(P)[x'_i(t) + D_i(P)u'_{i-1}(t-1)] \\ &\quad + B_i(P)[y'_i(t) + D_i(P)u'_{i-1}(t-1)] + C_i(P)u'_{i-1}(t-1) \\ &= \gamma_i + A_i(P)x'_i(t) + B_i(P)y'_i(t) + C'_i(P)u'_{i-1}(t-1), \end{aligned} \tag{33}$$

where

$$C'_i(P) := C_i(P) + A_i(P) \cdot D_i(P) + B_i(P) \cdot D_i(P). \tag{34}$$

These equations are based on an information set that does not include terms from the future:

$$\begin{aligned} \mathcal{I}'_i(t) &:= \{x'_i(t), x'_i(t-1), \dots, x'_i(-\infty); y'_i(t), y'_i(t-1), \dots, y'_i(-\infty); \\ &\quad u'_{i-1}(t-1), \dots, u'_{i-1}(t-h_i), \dots, u_{i-1}(-\infty)\}. \end{aligned}$$

Note that (33) and the new information set $\mathcal{I}'_i(t)$ have the same structure as (6) and its information set $\mathcal{I}_i(t)$. Therefore the new variables satisfy Theorem 3; i.e., the bullwhip effect exists

if

$$\begin{aligned} 0 < \sum_{i=1}^I \frac{1 + B_i(1)l_i - C'_i(1)}{A_i(1) + B_i(1)} &= \sum_{i=1}^I \frac{1 + B_i(1)l_i - [C_i(1) + h_i A_i(1) + h_i B_i(1)]}{A_i(1) + B_i(1)} \\ &= \sum_{i=1}^I \frac{1 + B_i(1)l_i - C_i(1)}{A_i(1) + B_i(1)} - \sum_{i=1}^I h_i. \end{aligned}$$

The first equality holds because (as we can see from (34)) $D_i(1) = h_i$ and

$$C'_i(1) = C_i(1) + h_i A_i(1) + h_i B_i(1), \forall i.$$

This proves Theorem 4.

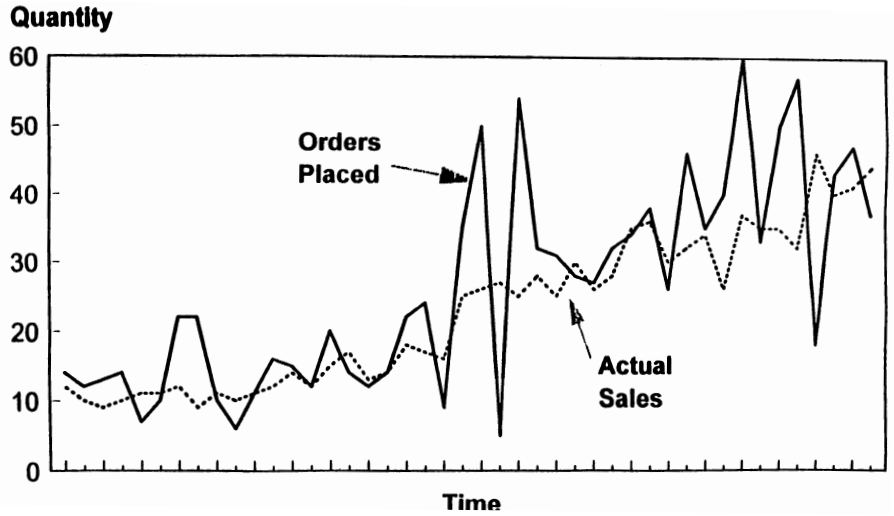


Figure 1: Amplification of order variability for one stage (Lee et al. [22])

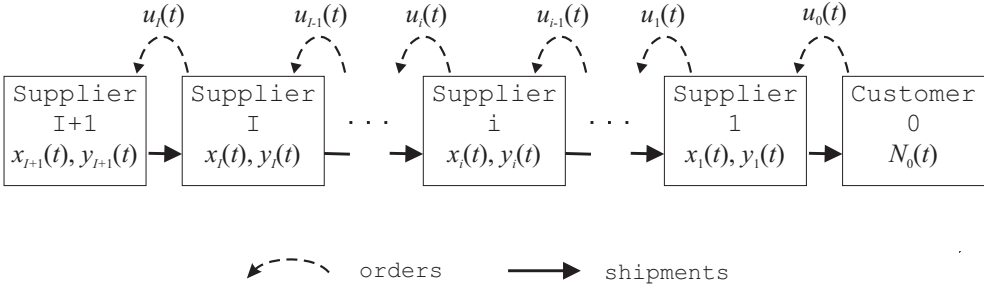


Figure 2: A supply chain representation

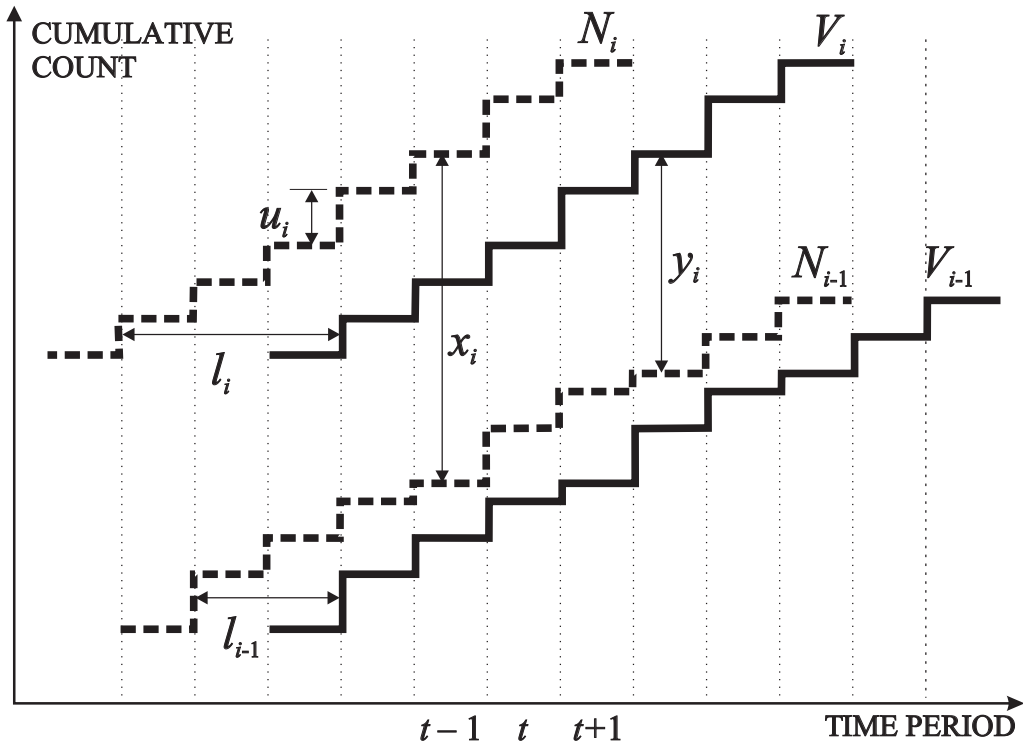


Figure 3: A representation of system operations

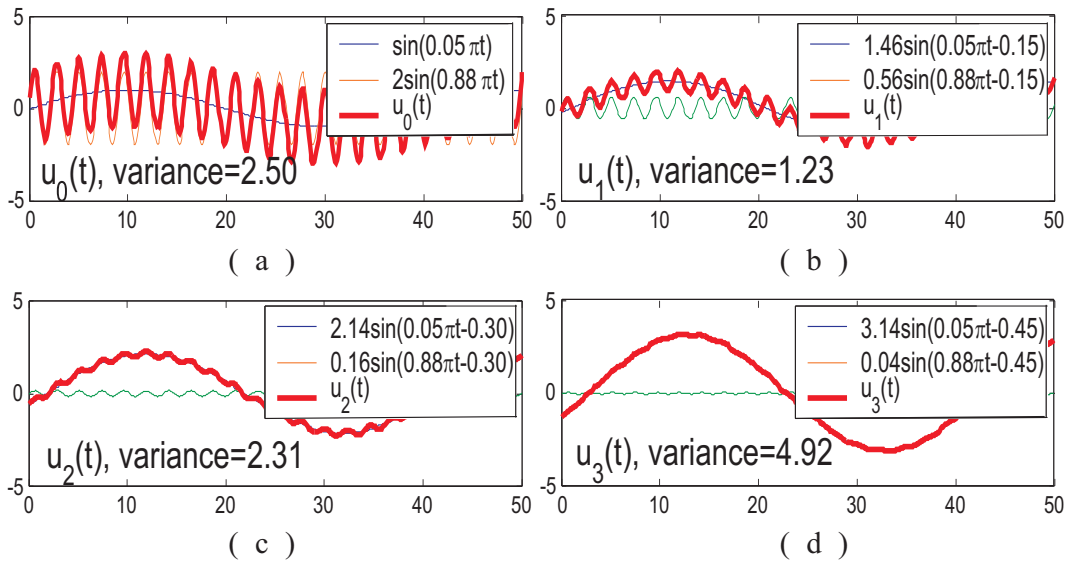
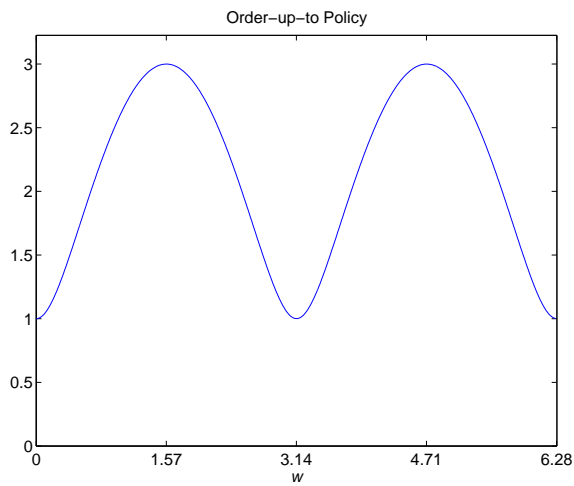
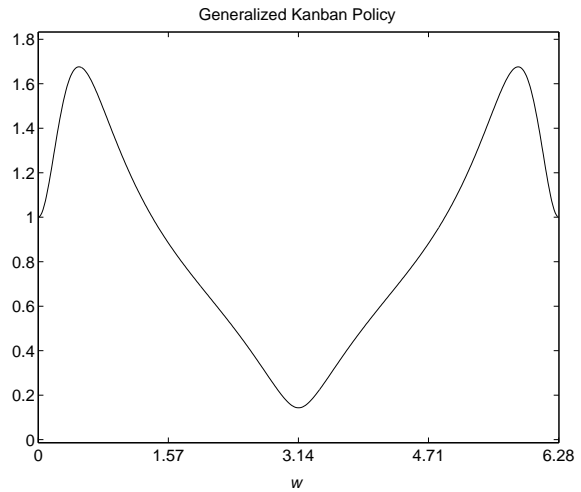


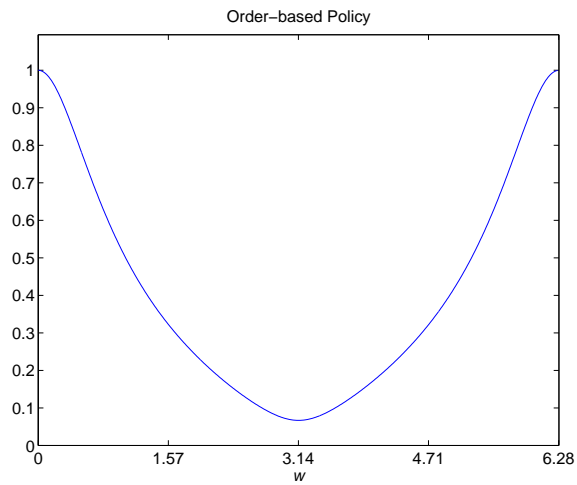
Figure 4: Variance amplification along a supply chain



(a)



(b)



(c)

Figure 5: Amplification factors for three policies: (a) Order-up-to policy; (b) Kanban policy; (c) Order-based policy.

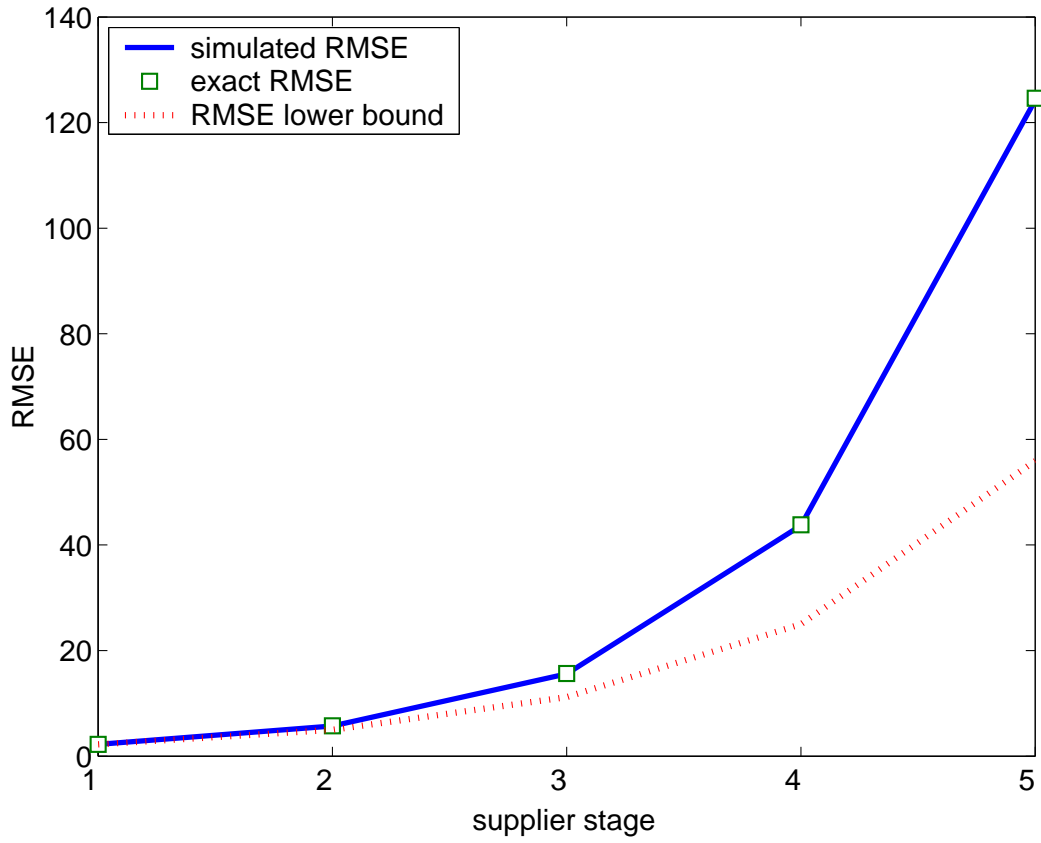
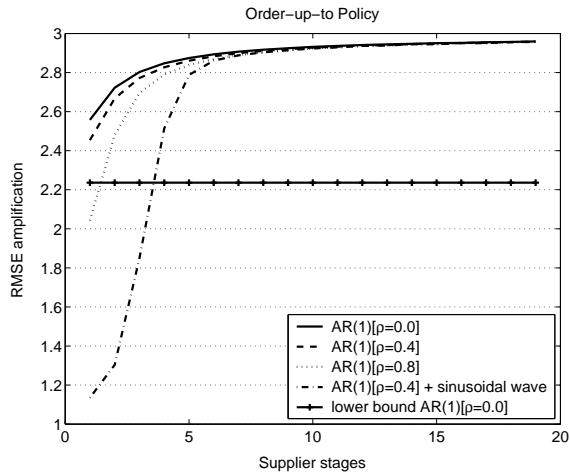
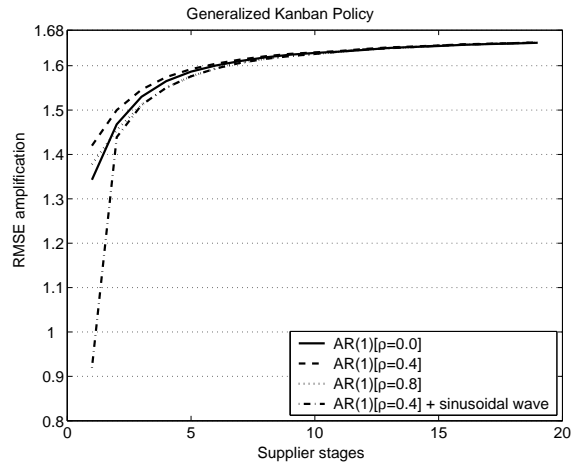


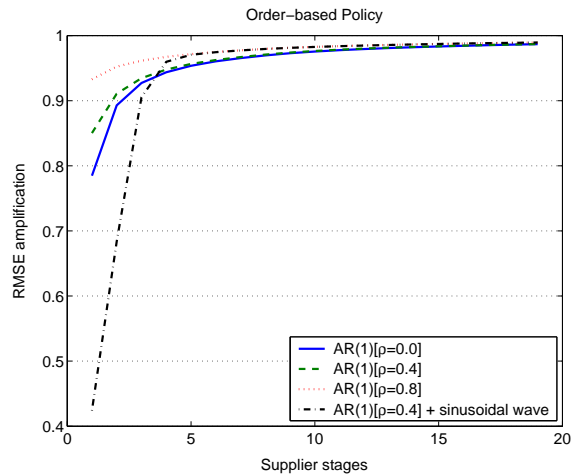
Figure 6: RMSE at each stage for the order-up-to policy



(a)

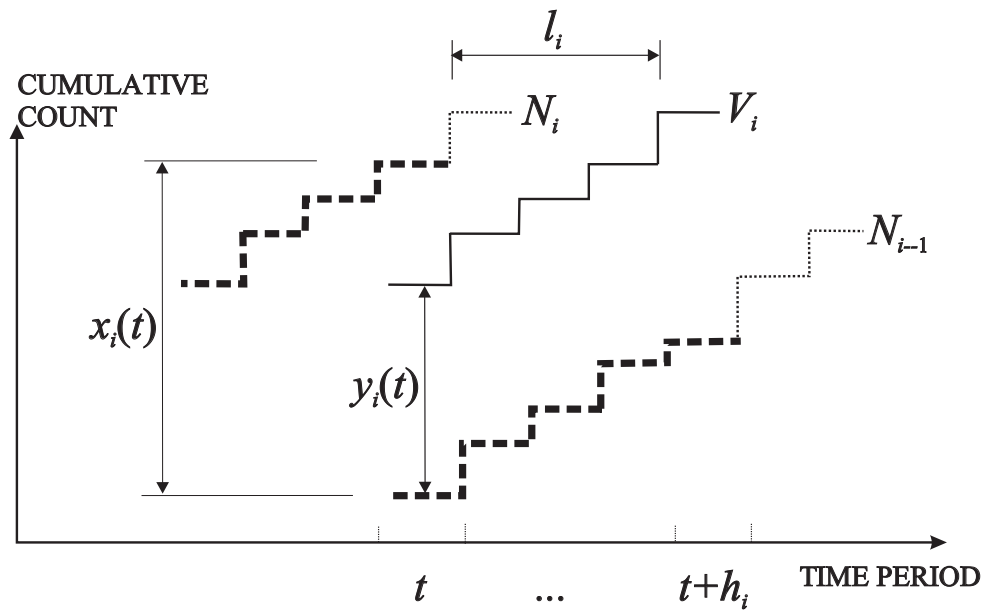


(b)

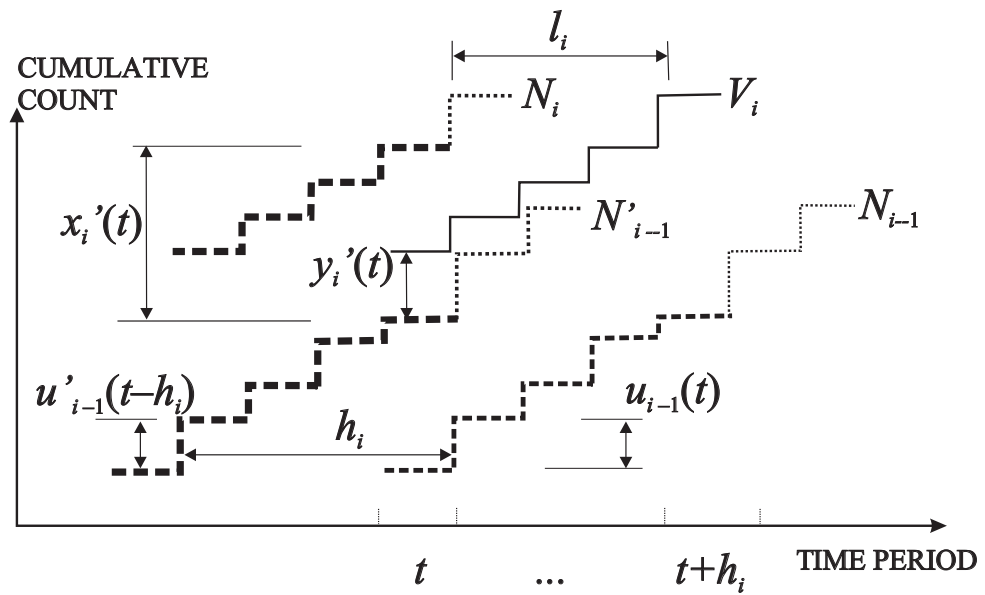


(c)

Figure 7: Amplification of root mean square error at each stage; (a) Order-up-to policy; (b) Kanban policy; (c) Order-based policy.



(a)



(b)

Figure 8: Effect of h -period ADI; (a) original system; (b) shifted system.