UNIVERSITY OF CALIFORNIA BERKELEY
Department of Civil Engineering
Fall 2010

Structural Engineering,
Mechanics and Materials
Professor: S. Govindjee

Adjugate

The adjugate[1] to a second order tensor $\boldsymbol{F}$ is defined to be the tensor $\boldsymbol{F}^*$ such that

$$(\boldsymbol{F}\cdot\boldsymbol{a})\times(\boldsymbol{F}\cdot\boldsymbol{b}) = \boldsymbol{F}^*\cdot(\boldsymbol{a}\times\boldsymbol{b}) \tag{1}$$

for all vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.[2]

This expression is useful in many derivations but does not tell us how to directly compute the adjugate given a tensor. Consider three arbitrary vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, then

$$\begin{aligned}
(\boldsymbol{F}\cdot\boldsymbol{c})\cdot[(\boldsymbol{F}\cdot\boldsymbol{a})\times(\boldsymbol{F}\cdot\boldsymbol{b})] &= (\boldsymbol{F}\cdot\boldsymbol{c})\cdot\boldsymbol{F}^*\cdot[\boldsymbol{a}\times\boldsymbol{b}] && (2)\\
J\boldsymbol{c}\cdot[\boldsymbol{a}\times\boldsymbol{b}] &= (\boldsymbol{F}\cdot\boldsymbol{c})\cdot\boldsymbol{F}^*\cdot[\boldsymbol{a}\times\boldsymbol{b}] && (3)\\
J\boldsymbol{F}^{-1}\boldsymbol{c}\cdot[\boldsymbol{a}\times\boldsymbol{b}] &= \boldsymbol{c}\cdot\boldsymbol{F}^*\cdot[\boldsymbol{a}\times\boldsymbol{b}] && (4)\\
\boldsymbol{c}\cdot J\boldsymbol{F}^{-T}\cdot[\boldsymbol{a}\times\boldsymbol{b}] &= \boldsymbol{c}\cdot\boldsymbol{F}^*\cdot[\boldsymbol{a}\times\boldsymbol{b}] && (5)\\
J\boldsymbol{F}^{-T}\cdot[\boldsymbol{a}\times\boldsymbol{b}] &= \boldsymbol{F}^*\cdot[\boldsymbol{a}\times\boldsymbol{b}] && (6)\\
J\boldsymbol{F}^{-T} &= \boldsymbol{F}^*\,. && (7)
\end{aligned}$$

Equation (2) uses the formal definition of the adjugate. In going to Eq. (3) we have used the definition of the determinant. The next step involves premulitplying both sides by $\boldsymbol{F}^{-1}$. To get to Eq. (5) the definition of the transpose has been applied. To get to Eq. (6) we note that $\boldsymbol{c}$ is arbitrary and to get to the working definition Eq. (7) we note that $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary and thus so is their cross-product.

[1]The adjugate is also the transpose of the matrix of co-factors.
[2]The adjugate is sometimes expressed with the notation $\mathrm{adj}(\boldsymbol{F})$.