UNIVERSITY OF CALIFORNIA BERKELEY | Structural Engineering,
Department of Civil Engineering | Mechanics and Materials
Fall 2008 | Professor: S. Govindjee

Michell's Solution to the Bi-Harmonic

For the case where stress in not a function of θ then the solution to

$$\nabla^4\Phi = \left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}\right)\left(\Phi_{,rr} + \frac{1}{r}\Phi_{,r} + \frac{1}{r^2}\Phi_{,\theta\theta}\right) = 0 \qquad (1)$$

is given by

$$\Phi = a_o'\theta + a_o \ln(r) + b_o r^2 + c_o r^2 \ln(r) \qquad (2)$$

If the coordinate system origin is in the body (is a material point) then $a_o' = c_o = a_o = 0$. If the body is multiply connected, then to ensure single valuedness the following integrability conditions on each internal cavity must be satisfied:

$$\int_{C_i} \frac{d}{dn}(\nabla^2\Phi)\, ds = 0 \qquad (3)$$

$$\int_{C_i} \left[y\frac{d}{ds}(\nabla^2\Phi) + x\frac{d}{dn}(\nabla^2\Phi)\right] ds = -\frac{1}{1-\nu}B_x \qquad (4)$$

$$\int_{C_i} \left[y\frac{d}{dn}(\nabla^2\Phi) - x\frac{d}{ds}(\nabla^2\Phi)\right] ds = -\frac{1}{1-\nu}B_y \qquad (5)$$

where d/ds and d/dn respectively denote tangential and normal derivatives to the cavity boundary and B_x and B_y are the body force components. The first of these conditions requires $a_o = 0$ if the origin is surrounded by an internal boundary.

The Michell solution to $\nabla^4\Phi = 0$ is given by consider a separable solution in the spirit of Levy as $f_n(r)\exp[\pm in\theta]$. The result of much algebra and the careful application of "variation of parameters" to treat multiple repeated roots is:

$$\begin{aligned}
\Phi &= a_o \ln(r) + b_o r^2 + c_o r^2 \ln(r) + d_o r^2 \theta + a'_o \theta && (6)\\
&+ a_1 r\theta \sin(\theta) + \big(b_1 r^3 + a'_1/r + b'_1 r \ln(r)\big) \cos(\theta) && (7)\\
&+ c_1 r\theta \cos(\theta) + \big(d_1 r^3 + c'_1/r + d'_1 r \ln(r)\big) \sin(\theta) && (8)\\
&+ \sum_{n=2}^{\infty} \big(a_n r^n + b_n r^{n+2} + a'_n/r^n + b'_n/r^{n-2}\big) \cos(n\theta) && (9)\\
&+ \sum_{n=2}^{\infty} \big(c_n r^n + d_n r^{n+2} + c'_n/r^n + d'_n/r^{n-2}\big) \sin(n\theta) && (10)
\end{aligned}$$

where a_i *etc.* are constants.
Remarks:

1. The term $c_o r^2 \ln(r)$ implies that u_θ is proportional to θ. Therefore, $c_o = 0$ if the origin is surrounded by any body boundary.

2. The term $d_o r^2 \theta$ implies that u_r is proportional to θ. Therefore, $d_o = 0$ if the origin is surrounded by any body boundary.

3. The terms $a_1 r\theta \sin(\theta)$ and $b'_1 r \ln(r) \cos(\theta)$ imply a multiple valued result if the origin is surrounded by any body boundary. It can be shown that this requires $b'_1 = -a_1(1-2\nu)/(2(1-\nu))$; further, if the origin is a material point then $a_1 = b'_1 = 0$.

4. The terms $c_1 r\theta \cos(\theta)$ and $d'_1 r \ln(r) \sin(\theta)$ imply a multiple valued result if the origin is surrounded by any body boundary. It can be shown that this requires $d'_1 = c_1(1-2\nu)/(2(1-\nu))$; further, if the origin is a material point then $c_1 = d'_1 = 0$.

5. The solution above comes from a paper by J.H. Michell, *Proc. London Math Soc.*, vol. 31, p.100, 1899. A disscusion on the use of this solution can be found in *Theory of Elasticity* by S.P. Timoshenko and J.N. Goodier, Art. 43-46. A more comprehensive discussion of the properties of the solution to the biharmonic equation in the context of elasticity may be found in *Mathematical Theory of Elasticity* by I.S. Sokolnikoff, Art. 69-70. The essense of the art of using the general solution is to understand through the computation of examples what the individual terms do. Then to consider their linear combinations in such a way as to solve practical problems of interest.

6. Note that at times the general solution can be a bit too general. For instance, while expressions of the form $\theta \ln(r)$ and $r^2\theta \ln(r)$ are formally part of the solution through the approriate setting of inter-relations between the coefficients, it can require the computation of many terms for reasonable convergence. Thus it can be advantageous to approach the solution to the bi-harmonic equation through separable expansions of the form $\theta f(r)$.