UNIVERSITY OF CALIFORNIA BERKELEY — Structural Engineering, Mechanics and Materials
Department of Civil Engineering — Professor: S. Govindjee
Fall 2002

Elastic-Perfectly Plastic Thick Walled Sphere

Consider a thick walled sphere made of an elastic perfectly plastic material that is governed by von Mises' yield condition. Assume the sphere is subject to an internal pressure p_i and an external pressure p_o. Since the problem is spherically symmetric there is only one non-trivial equilibrium equation

$$\tau_{rr,r} = \frac{2}{r}(\tau_{\theta\theta} - \tau_{rr})\,. \tag{1}$$

Note that

$$\tau_{\varphi\varphi} = \tau_{\theta\theta} \tag{2}$$

by symmetry. In this case the strain displacement relations reduced down to

$$\begin{aligned} \varepsilon_{rr} &= u_{r,r} && (3)\\ \varepsilon_{\theta\theta} &= \varepsilon_{\varphi\varphi} = u_r/r\,. && (4) \end{aligned}$$

The relevant elastic constitutive relations are

$$\begin{aligned} \varepsilon_{rr} &= \frac{1}{E}[\tau_{rr} - 2\nu\tau_{\theta\theta}] && (5)\\ \varepsilon_{\theta\theta} &= \frac{1}{E}[(1-\nu)\tau_{\theta\theta} - \nu\tau_{rr}]\,. && (6) \end{aligned}$$

A compatibility equation[1] in this setting can derived by combining (3) and (4) to give

$$\varepsilon_{\theta\theta,r} = (\varepsilon_{rr} - \varepsilon_{\theta\theta})/r\,. \tag{7}$$

The boundary conditions maybe expressed as

$$\begin{array}{rclccl} \boldsymbol{\tau}\boldsymbol{e}_r & = & -p_o\boldsymbol{e}_r & \text{at} & r = r_o & \\ \boldsymbol{\tau}(-\boldsymbol{e}_r) & = & p_i\boldsymbol{e}_r & \text{at} & r = r_i\,. & \end{array} \tag{8}$$

[1] Relation between strains eliminating displacements but still guaranteeing that the strains are the symmetric gradient of a single-valued displacement field

Since we have pure traction boundary conditions, this problem can be attacked as a pure stress problem.

Step 1: Begin by deriving a simplified Beltrami-Mitchell equation[2]. Plug (5) and (6) into (7), and use the facts from equilibrium that $(\tau_{rr} - \tau_{\theta\theta})/r = \frac{1}{2}\tau_{rr,r}$ and $\tau_{\theta\theta,r} = (\frac{r}{2}\tau_{rr,r} + \tau_{rr})_{,r}$. The net result is that

$$\begin{aligned} 0 &= r\tau_{rr,rr} + 4\tau_{rr,r} \\ &= (r\tau_{rr,r})_{,r} + 3\tau_{rr,r} \, . \end{aligned} \tag{9}$$

Step 2: Solve the compatibility equation (9) for the radial stress. Integrating twice gives

$$\tau_{rr} = C + Dr^{-3} \, , \tag{10}$$

where C and D are constants of integration.

Step 3: Use equilibrium to get the hoop stresses.

$$-3Dr^{-4} = \frac{2}{r}[\tau_{\theta\theta} - Dr^{-3} - C] \, ;$$

which implies that

$$\tau_{\theta\theta} = \tau_{\varphi\varphi} = C - \frac{D}{2}r^{-3} \, . \tag{11}$$

Step 4: Apply the boundary conditions to determine the constants.

$$\begin{aligned} C + D/r_i^3 &= -p_i \\ C + D/r_o^3 &= -p_o \, . \end{aligned}$$

Solving for C and D gives

$$\begin{aligned} C &= \frac{p_o r_o^3 - p_i r_i^3}{r_i^3 - r_o^3} \\ D &= -(p_i - p_o)\frac{r_i^3 r_o^3}{r_o^3 - r_i^3} \, . \end{aligned} \tag{12}$$

Step 5: Determine the displacements by using (6) and (4). This gives

$$u_r = r\varepsilon_{\theta\theta} = \frac{r}{E}(\tau_{\theta\theta}(1-\nu) - \nu\tau_{rr}) \, .$$

[2] The compatibility equation written in terms of stresses

Inserting from above for the stresses gives

$$u_r = \frac{r}{E}\left[(1-2\nu)C + \frac{1-3\nu}{2}Dr^{-3}\right]. \tag{13}$$

Step 6: Determine the condition for yield. von Mises' condition in this case reduces to

$$\frac{1}{2}\{(\tau_{rr}-\tau_{\theta\theta})^2 + (\tau_{rr}-\tau_{\varphi\varphi})^2 + (\tau_{\theta\theta}-\tau_{\varphi\varphi})^2\} \le \tau_Y^2\,.$$

If we plug in the elastic expressions for the stress from above we see that

$$\left(\frac{3}{2}Dr^{-3}\right)^2 \le \tau_Y^2\,.$$

Plugging in $(12)_2$ gives

$$\frac{|p_i-p_o|}{r^3} \le \frac{2}{3}\tau_Y\frac{r_o^3-r_i^3}{r_i^3r_o^3}\,.$$

This expression show that yielding will first occur on the inner wall of the sphere. Therefore the yielding will first occur when

$$|p_i-p_o| = \frac{2}{3}\tau_Y\frac{r_o^3-r_i^3}{r_o^3}\,. \tag{14}$$

Step 7: Suppose, now, yielding has progressed to $r = R$. The region $r \ge R$ is still elastic; therefore, the original solution still holds but with modified boundary conditions

$$\tau_{rr} = -p_o \qquad \text{at} \quad r = r_o$$

and

$$\tau_{\theta\theta} - \tau_{rr} = \tau_Y \qquad \text{at} \quad r = R\,.$$

This last expression is the condition on the stresses when yield is occuring. If we resolve for the constants C and D, then we find that

$$C = -p_o + \frac{2}{3}\tau_Y\left(\frac{R}{r_o}\right)^3 \tag{15}$$

$$D = -\frac{2}{3}\tau_Y R^3 \tag{16}$$

Thus in the region $r \geq R$

$$
\begin{aligned}
\tau_{rr} &= -p_o + \frac{2}{3}\tau_Y \left(\frac{R}{r}\right)^3 \left[\left(\frac{r}{r_o}\right)^3 - 1\right] \\
\tau_{\theta\theta} &= -p_o + \frac{2}{3}\tau_Y \left(\frac{R}{r}\right)^3 \left[\left(\frac{r}{r_o}\right)^3 - \frac{1}{2}\right] .
\end{aligned}
$$

To determine the strains in this region simply apply the constitutive relations. For the displacements in this region, note that Eq. (13) still holds using expression (15) and (16) for the constants.

Step 8: To determine the stresses in the plastic zone, first note that the von Mises condition tells us that for states on the yield surface that

$$
\tau_{\theta\theta} - \tau_{rr} = \tau_Y .
$$

If we plug this into the equilibrium equation then we have in the region $r \leq R$ that

$$
\tau_{rr,r} = 2\tau_Y / r .
$$

Solving this equation gives

$$
\tau_{rr} = 2\tau_Y \ln(r) + C .
$$

At the inner radius the boundary condition states that $\tau_{rr} = -p_i$. Thus, we have that $C = -p_i - 2\tau_Y \ln(r_i)$. This gives the stress state in the region $r \leq R$ as

$$
\begin{aligned}
\tau_{rr} &= 2\tau_Y \ln\left(\tfrac{r}{r_i}\right) - p_i \\
\tau_{\theta\theta} &= \tau_{\varphi\varphi} = \tau_Y \left[1 + 2\ln\left(\tfrac{r}{r_i}\right)\right] - p_i .
\end{aligned}
\tag{17}
$$

Note that the last expression comes from the von Mises expression; do not forget that perfect plasticity is being assumed and that the pressure differential is being assumed to be monotonically increasing.

Step 9: Find the value of R for a given pressure differential. This is done by requiring traction continuity at the interface between elastic and plastic regions. In the present case the normal to the interface between the regions is simply $\boldsymbol{e}_r$. Thus, at $r = R$ we must have

$$
\boldsymbol{\tau}_{elastic}\boldsymbol{e}_r = \boldsymbol{\tau}_{plastic}\boldsymbol{e}_r .
$$

Since all the shear stresses are zero in this problem, we simply need to match τ_{rr} at the interface. This gives

$$2\tau_Y \ln\left(\frac{R}{r_i}\right) - p_i = -p_o + \frac{2}{3}\tau_Y \left(\frac{R}{R}\right)^3 \left[\left(\frac{R}{r_o}\right)^3 - 1\right].$$

Thus, we can say that

$$p_i - p_o = 2\tau_Y \left[\ln\left(\frac{R}{r_i}\right) + \frac{1}{3}\left(1 - \left(\frac{R}{r_o}\right)^3\right)\right] \tag{18}$$

is a nonlinear equation that determines the value of R for a given pressure differential. Note that the sphere is fully plastic where

$$p_i - p_o = 2\tau_Y \ln\left(\frac{r_o}{r_i}\right). \tag{19}$$

Step 10: To determine the displacements u_r for the region $r \leq R$ is a bit more complex. However since plasticity (J_2) only occurs in the deviatoric portion of the response we can take advantage of the fact that the bulk response is still elastic. Thus, we still have in this region that

$$(\varepsilon_{rr} + 2\varepsilon_{\theta\theta}) = \frac{1}{3K}(\tau_{rr} + 2\tau_{\theta\theta}).$$

We can now apply the strain-displacement relations and (17) to give the ordinary differential equation

$$\frac{1}{r^2}(r^2 u_r)_{,r} = \frac{2\tau_Y}{3K}(1 + 3\ln(r/r_i) - (3/2)p_i/\tau_Y).$$

This ODE can be solved for the displacement field. Note that in integrating this expression, the constant of integration can be eliminated by enforcing continuity of displacement at the interface between the elastic and plastic zones.