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The Symmetric Identity: $\mathbb{I}^{\text{sym}} \rightarrow \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

The derivative of a second order tensor with itself is the fourth order identity tensor $\mathbb{I}$. In component form this reads, for a tensor $\boldsymbol{A}$ with components A_{ij}, as

$$\mathbb{I}_{ijkl} = \frac{\partial A_{ij}}{\partial A_{kl}} = \delta_{ik}\delta_{jl}\,; \tag{1}$$

i.e. if $i = k$ *and* $j = l$ then the result is unity and if one of the equalities is violated, then the result is zero. So, for example, $\partial A_{12}/\partial A_{12} = 1$ and $\partial A_{12}/\partial A_{21} = 0$ etc.

When the tensor $\boldsymbol{A} \in \mathbb{S}$ (the set of symmetric tensors), then we come to the result that the identity is no longer given by (1). Rather it is given component-wise by the expression in the header. This then returns the somewhat non-intuitive result that $\partial A_{12}/\partial A_{12} = \mathbb{I}^{\text{sym}}_{1212} = 1/2$ (not unity!) even though $\partial A_{11}/\partial A_{11} = \mathbb{I}^{\text{sym}}_{1111} = 1$. How can this be?

Consider a function $f : \mathbb{S} \rightarrow \mathbb{R}$. Its derivative[1] is a linear mapping $Df : \mathbb{S} \rightarrow \mathbb{R}$. As such we can represent it by a tensor:

$$\boldsymbol{B} : \boldsymbol{C} = Df[\boldsymbol{C}]\,. \tag{2}$$

The tensor $\boldsymbol{B}$ characterizing the derivative of $f(\cdot)$ in the (symmetric) direction $\boldsymbol{C}$ is non-unique since one can add an arbitrary skew-symmetric tensor to $\boldsymbol{B}$ without changing the result. The symmetric part of $\boldsymbol{B}$, viz. $\frac{1}{2}(\boldsymbol{B}+\boldsymbol{B}^T)$, is however unique. This is the tensor which we *choose* to use to represent the derivative Df. We do this for two reasons: (1) it is unique, thus there is no ambiguity, and (2) it will always produce the correct result for the rate of change of $f(\cdot)$ in a *symmetric* direction. Note it only makes sense to discuss rates of change of $f(\cdot)$ in symmetric directions, since $f(\cdot)$ is only defined over the space of symmetric tensors.

Let us now apply this result to the component function $f(\boldsymbol{A}) = (\hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j) : \boldsymbol{A} = A_{ij}$. To compute the derivative of $f(\cdot)$ we apply the directional

[1] Assuming it exists

derivative formula

$$\left.\frac{d}{d\alpha}\right|_{\alpha=0} (\hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j) : (\boldsymbol{A} + \alpha \boldsymbol{H}) = Df[\boldsymbol{H}]\,, \tag{3}$$

where $\boldsymbol{H} \in \mathbb{S}$. Taking the derivative with respect to α, setting α to zero, leads to:

$$(\hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j) : \boldsymbol{H} = Df[\boldsymbol{H}]\,, \tag{4}$$

which allows us to identify *an* expression for $\boldsymbol{B}$ as $\hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j$. The unique part of $\boldsymbol{B}$ which provides the derivative information is the symmetric part; viz. $\frac{1}{2}(\hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j + \hat{\boldsymbol{e}}_j \otimes \hat{\boldsymbol{e}}_i)$. This expression gives us the derivative $\partial A_{ij}/\partial \boldsymbol{A}$. If we now compute its components we get

$$\frac{\partial A_{ij}}{\partial A_{kl}} = \frac{\partial A_{ij}}{\partial \boldsymbol{A}} : (\hat{\boldsymbol{e}}_k \otimes \hat{\boldsymbol{e}}_l) = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,. \tag{5}$$

Thus we arrive at the desired expression – an expression for the fourth order identity tensor over the space of symmetric tensors. Observe that this expression yields the results $\partial A_{11}/\partial A_{11} = \mathbb{I}^{\mathrm{sym}}_{1111} = 1$, $\partial A_{12}/\partial A_{12} = \mathbb{I}^{\mathrm{sym}}_{1212} = 1/2$, as well as $\partial A_{12}/\partial A_{21} = \mathbb{I}^{\mathrm{sym}}_{1221} = 1/2$.