UNIVERSITY OF CALIFORNIA, BERKELEY Dept. of Civil and Environmental Engineering
FALL SEMESTER 2018 Structural Engineering, Mechanics and Materials

NAME ____________________________

PH.D. PRELIMINARY EXAMINATION

MATHEMATICS

Problem 1. (40 points)

Define

$$G(t) = \int_0^\infty \Big(\int_{-\infty}^{\frac{t}{\sqrt{\nu}}\sqrt{u}} \frac{(1/2)^{\nu/2}}{\sqrt{2\pi}\Gamma(\nu/2)} u^{\nu/2-1} \exp(-\frac{z^2+u}{2}) dz \Big) du \tag{1}$$

where ν is a constant, and $\Gamma(\nu/2)$ is Gamma function, which you do not need to evaluate. Just leave it there.

Calculate or find the expression for

$$g(t) = \frac{dG(t)}{dt} \ ?$$

Hint: Apply the fundamental theorem of calculus and chain rule. You do not need to integrate u.

Problem 2 (60 points)

Consider a smooth function $f(x) \geq 0$, and

$$\frac{df}{dx} = -\frac{f^2(x)}{1-F(x)} =: f'(x)$$

where $1 \geq F(x) = \int_{-\infty}^x f(t)dt > 0$ or $F'(x) = \dfrac{dF}{dx} = f(x)$.

(1) Calculate $f''(x)$, $f'''(x) \cdots$ and verify that

$$\frac{d^k f}{dx^k}(x) = (-1)^k \frac{f^{k+1}(x)}{(1-F(x))^k}, \quad k = 1, 2, \cdots$$

i.e. assume that

$$\frac{d^k f}{dx^k}(x) = (-1)^k \frac{f^{k+1}(x)}{(1-F(x))^k}, \quad k = 1, 2, \cdots$$

show that

$$\frac{d^{k+1} f}{dx^{k+1}}(x) = (-1)^k \frac{f^{k+2}(x)}{(1-F(x))^{k+1}}, \quad k = 1, 2, \cdots$$

(2) Let

$$F_n(x) := [F(x)]^n \ , \ n = 1, 2, \cdots$$

Find

$$f_n(x) := \frac{dF_n}{dx} = ?$$

and show that

$$f_n'(x) = \frac{d^2 F_n}{dx^2} = n f'(x)[F(x)]^{n-1} + n(n-1) f^2(x)[F(x)]^{n-2}$$

(3) Assume that at $x = x_n$, $f_n'(x_n) = 0$, find

$$F(x_n) = \ ?$$

(4) Assume that

$$\frac{f(x_n)}{(1 - F(x_n)} = \alpha_n = const. \quad \rightarrow \quad \text{Find } f(x_n) \ ?$$

(5) Consider the Taylor series expansion of $F(x)$ at $x = x_n$, i.e.

$$F(x) = F(x_n) + F'(x)(x - x_n) + \frac{1}{2!} F''(x_n)(x - x_n)^2 + \frac{1}{3!} F'''(x_n)(x - x_n)^3 + \cdots$$

Show that

$$F(x) = 1 - \frac{1}{n}\Big[1 - \frac{\alpha_n (x - x_n)}{1!} + \frac{\alpha_n^2 (x - x_n)^2}{2!} - \frac{\alpha_n^3 (x - x_n)^3}{3!} + \cdots\Big]$$

and subsequently

$$F(x) = 1 - \frac{1}{n} \exp(-\alpha_n (x - x_n)) \ .$$

NAME ___________________________

PH.D. PRELIMINARY EXAMINATION

MATHEMATICS

Problem 1. (40 points)

Define an average operator in **R** as

$$< f > (x) := \int_{-\infty}^{\infty} f(y) \exp(-a(x-y)^2) dy, \text{ where } |f(y)| < c, \forall y \in \mathbf{R}$$

where $a > 0$ and $0 < c < \infty$ are real numbers.
Show that

$$\frac{d}{dx} < f > (x) = < \frac{df}{dy} > \ . \tag{1}$$

Problem 2. (40 points)

Consider the following differential equation,

$$EI \frac{d^4 v}{dx^4} = q(x), \quad \forall\ 0 < x < L \tag{2}$$

where $g(x)$ is a given function, and the differential equation has the following boundary conditions:

$$v(0) = 0, \quad v'(0) = 0, \quad EIv''(L) = \bar{M}, \ EIv'''(L) = \bar{V}$$

Consider a given function $w(x)$ with the boundary conditions

$$w(0) = 0, \quad w'(0) = 0\ , \quad w(L) = 1, \quad w'(L) = -1.$$

Evaluate the following definite integral,

$$\int_0^L EIv''(x)w''(x)dx = ?$$

where $v' = \frac{dv}{dx}$, $v'' = \frac{d^2 v}{dx^2}$ and $v''' = \frac{d^3 v}{dx^3}$.

Problem 3. (20 points)

Consider the following algebraic equation,

$$\begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{bmatrix} N_1(x) \\ N_2(x) \end{bmatrix}$$

where $N_1(x)$ and $N_2(x)$ are unknown functions, and x_1, x_2 are two given points in the real number axis **R**. Find $N_1(x)$ and $N_2(x)$?
Under which condition, $N_1(x)$ and $N_2(x)$ do not exit.

UNIVERSITY OF CALIFORNIA, BERKELEY Dept. of Civil and Environmental Engineering
FALL SEMESTER 2017 Structural Engineering, Mechanics and Materials

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PH.D. PRELIMINARY EXAMINATION

MATHEMATICS

Problem 1. (40 points)

Consider a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

where a, b, c and d are real numbers, and $\mathbf{I}$ is the 2×2 unit matrix.

Define the characteristic equation of $\mathbf{A}$ as,

$$p(\lambda) = det(\lambda \mathbf{I} - \mathbf{A}) = 0, \tag{2}$$

where λ is its eigenvalue, i.e.

$$\mathbf{AX} = \lambda \mathbf{X}, \quad \mathbf{X} \neq 0, \text{and} \quad \mathbf{X} \in \mathbb{R}^2$$

Show that

$$p(\mathbf{A}) = \mathbf{0}. \tag{3}$$

Problem 2. (60 points)

Consider the following nonlinear differential equation,

$$\frac{1}{\left(1 + (y')^2\right)^{3/2}} \frac{d^2 y}{dx^2} = \kappa, \quad \forall\ 0 < x < L \tag{4}$$

where $y' = \frac{dy}{dx}$ and $\kappa = const.$ with the following boundary conditions:

$$y(0) = 0, \quad y'(0) = 0.$$

Find the solution $y(x)$.

Hint: Let $y' = \tan \theta$.

UNIVERSITY OF CALIFORNIA, BERKELEY — Dept. of Civil and Environmental Engineering
SPRING SEMESTER 2017 — Structural Engineering, Mechanics and Materials

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PH.D. PRELIMINARY EXAMINATION

MATHEMATICS

Problem 1 (40 points)
Consider the following system of ordinary differential equations

$$\frac{dx}{dt} = 3x - 4y, \quad \frac{dy}{dt} = 4x - 7y \tag{1}$$

which are subjected to the following initial condition,

$$x(0) = y(0) = 1 \ . \tag{2}$$

(1) Write the system of equation in a form

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}, \quad \text{where} \ \ \mathbf{x} = (x, y)^T;$$

(2) Find the eigenvalue of $\mathbf{A}$;

(3) Find the corresponding eigenvectors;

(4) Find the complete solution of the above ODEs by using the initial conditions.

Problem 2 (60 points)

Consider the following interpolation function,

$$v(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3, \quad 0 \le x \le 1$$

where c_0, c_1, c_2 and c_3 are unknown constants.
Assume that

$$\begin{aligned} v(0) &= u_1 \\ v'(0) &= u_2 \\ v(1) &= u_3 \\ v'(1) &= u_4 \end{aligned}$$

Express $v(x)$ in terms of u_1, u_2, u_3 and u_4.

UNIVERSITY OF CALIFORNIA, BERKELEY   Dept. of Civil and Environmental Engineering
FALL SEMESTER 2016   Structural Engineering, Mechanics and Materials

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PH.D. PRELIMINARY EXAMINATION (MATHEMATICS)

Problem 1. (50 points)
Consider the following nonlinear ordinary differential equation,

$$\frac{1}{(1+(y')^2)^{3/2}} y'' = \frac{1}{R} = const., \tag{1}$$

where $y' := \dfrac{dy}{dx}$ and $y'' := \dfrac{d^2y}{dx^2}$.

Solve this differential equation:
(1) Find $y'(x)$ with the boundary condition $y'(0) = 0$;
(2) Find $y(x)$ with the boundary condition $y(0) = R$.

Problem 2 (50 points)
Consider the following fourth order ordinary differential equation of an elastic beam-column,

$$\frac{d^4w}{dx^4} + \lambda^2 \frac{d^2w}{dx^2} = 0, \quad 0 < x < L \tag{2}$$

with boundary conditions,

$$w(0) = w'(0) = w(L) = w'(L) = 0 \ . \tag{3}$$

Find:
(1) A trivial solution of Eqs. (2) and (3);

(2) The conditions for the existence of non-trivial solutions of Eqs. (2) and (3) .

Hints:
The general solution of Eq. (2) has the following form:

$$w(x) = A\cos\lambda x + B\sin\lambda x + Cx + D,$$

where A, B, C and D are unknown constants, which you do not need to determine XXX explicitly.

Mathematics
PhD Preliminary Spring 2016

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1. (25 points) Consider a Hermitian matrix $\boldsymbol{A}$ (i.e. $\bar{\boldsymbol{A}}^T = \boldsymbol{A}$, where the superposed bar implies complex conjugation)

   (a) Prove that the eigenvalues of $\boldsymbol{A}$ are real.

   (b) Show that the eigenvectors of $\boldsymbol{A}$ are orthogonal for distinct eigenvalues.

2. (25 points) Consider a parametric surface

$$S = \{\boldsymbol{r}(u, v) \in \mathbb{R}^3 \mid \boldsymbol{r} = au^2\boldsymbol{e}_x + bv^3\boldsymbol{e}_y + cuv\boldsymbol{e}_z, \quad u \in (0, 1),\ v \in (0, 1)\} \tag{1}$$

   where u, v are non-dimensional parameters and a, b, c are constants with dimensions of length.

   (a) Find the normal vector field of the surface.

   (b) Find the surface area of S.

3. (50 points) Consider the 2π-periodic function $f(t + 2\pi) = f(t)$ where

$$f(t) = \begin{cases} h & 0 < t \leq \pi \\ 0 & \pi < t \leq 2\pi \end{cases} \tag{2}$$

   (a) Find the Fourier series representation of $f(t)$.

   (b) What is the value of this representation at $x = \pi$? Does this make sense?

   (c) Consider the ordinary differential equation

$$m\ddot{x} + kx = f(t) \tag{3}$$

   with initial conditions $x(0) = 0$, $\dot{x}(0) = 0$. Find $x(t)$.

Mathematics
PhD Preliminary Exam Fall 2015

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1. (10 points) Solve the follow partial differential equation for $u(x,t)$:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad x \in (0,l)$$

Subject to the boundary conditions $u(0,t) = u(l,t) = 0 \ \forall t \geq 0$ and initial condition $u(x,0) = \sin(3\pi x/l)$ for $x \in (0,l)$.

2. (10 points) Solve the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ for $\boldsymbol{x}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

3. (10 points) The n^{th} order homogeneous linear differential equation with constant coefficients

$$\sum_{k=0}^{n} a_k \frac{d^k y}{dx^k} = 0$$

admits solutions of the form $y(x) = e^{sx}$. Find the form of the solution, $y(x)$, to the homogeneous Cauchy-Euler equation

$$\sum_{k=0}^{n} x^k \frac{d^k y}{dx^k} = 0 \qquad x > 0$$

by employing a change of variables from x to u that is defined by $x = e^u$.

NAME ________________________

PH.D. PRELIMINARY EXAMINATION

MATHEMATICS

Problem 1 (40 points)
Consider a square 3×3 real number matrix $\mathbf{A}$, whose eigenvalue problem is given as

$$\mathbf{A} = \lambda \mathbf{I}, \tag{1}$$

where $\mathbf{I}$ is the square unit matrix.

One can find its eigenvalues by solving the following characteristic polynomial equation,

$$\det[\mathbf{A} - \lambda \mathbf{I}] = -\lambda^3 + I_1 \mathbf{A})\lambda^2 - I_2(\mathbf{A})\lambda + I_3(\mathbf{A}) = 0, \qquad (1) \tag{2}$$

where

$$I_1(\mathbf{A}) = A_{11} + A_{22} + A_{33}, \quad I_2(\mathbf{A}) = \frac{1}{2}\Big(I_1^2(\mathbf{A}) - I_1(\mathbf{A}^2)\Big), \text{ and } I_3 = \det(\mathbf{A}) \ .$$

Show that the following matrix equation holds

$$-\mathbf{A}^3 + I_1(\mathbf{A})\mathbf{A}^2 - I_2(\mathbf{A})\mathbf{A} + I_3(\mathbf{A})\mathbf{I} = 0 \ , \qquad (2)$$

which is the so-called Cayley-Hamilton theorem.

Hint:
Multiply the equation (1) with the second order identity matrix $\mathbf{I}$.

Problem 2 (60 points)

Define an integration as

$$I(y(x)) = \int_0^\ell \sqrt{1 + (y'(x))^2} dx, \quad \text{where} \quad y' = \frac{dy}{dx}$$

where $y(x)$ is a smooth real function defined in $[0, \ell]$. Obviously, the value of integration depends on the selection of the function $y(x)$ (In fact, it is a functional, but you do not need that knowledge to solve the problem).

Let

$$I(y(x) + \epsilon w(x)) = \int_0^\ell \sqrt{1 + (y'(x) + \epsilon w'(x))^2} dx \ ,$$

where $\epsilon > 0$ is a real number, and $w(x)$ is another real function.

(1) For given functions $y(x)$ and $w(x)$, calculate the following quantity by taking the limit on ϵ,

$$\delta I := \lim_{\epsilon \to 0} \frac{1}{\epsilon}\Big(I(y(x) + \epsilon w(x)) - I(y(x))\Big)$$

(2) Define a real function: $f(\epsilon) := I(y(x) + \epsilon w(x))$. Expand it into the Taylor series of ϵ at $\epsilon = 0$ for the first three terms, i.e.

$$f(\epsilon) = f(0) + f'(0)\epsilon + (1/2)f^{"}(0)\epsilon^2 + o(\epsilon^2) \ .$$