Mechanics
PhD Preliminary Fall 2018

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1. (10 points) Consider the problem of small strain, isotropic, linear elasticity for two-dimensional plane strain. Show that the compatibility equation given in terms of the Airy stress function can be written as

$$\nabla^4 \phi = 0\,, \tag{1}$$

where in Cartesian form $\nabla^4\phi = \phi_{,1111} + 2\phi_{,1212} + \phi_{,2222}$. Recall also by definition that

$$\begin{aligned} \sigma_{11} &= \phi_{,22} && (2)\\ \sigma_{22} &= \phi_{,11} && (3)\\ \sigma_{12} &= -\phi_{,12}\,, && (4) \end{aligned}$$

and $\nabla \times \nabla \times \boldsymbol{\epsilon} = 0$, which in Cartesian form is given as $S_{pq} = e_{pki}e_{qlj}\epsilon_{ij,kl} = 0$.

2. (10 points) For the motion

$$\boldsymbol{x} = X_1\boldsymbol{e}_1 + (bX_1 + X_2)\boldsymbol{e}_2 + (X_3 + \frac{1}{2}c(X_3)^2)\boldsymbol{e}_3 \tag{5}$$

with given constants b and c, determine (i) the deformation gradient field $\boldsymbol{F}(\boldsymbol{X})$, (ii) the extremal stretches at the point $(X_1, X_2, X_3) = (1, 1, 1)$, and (iii) the volumetric strain as this point.

3. (10 points) Consider a material point whose stress state is given by

$$\boldsymbol{\sigma}(t) \sim \begin{bmatrix} 0 & a\cdot t & 0 \\ a\cdot t & 0 & 0 \\ 0 & 0 & a\cdot t \end{bmatrix}. \tag{6}$$

Assume the material to be a ductile metal governed by the Mises yield condition $f(\boldsymbol{\sigma}) = \|\boldsymbol{\sigma}'\| - \sqrt{\frac{2}{3}}\sigma_Y$, where $\boldsymbol{\sigma}'$ denotes the stress deviator and σ_Y is the material's yield stress. (i) Find an expression for the time of initial yield; (ii) Assuming associative plastic flow (Prandtl-Reuss), determine the direction of plastic flow at the moment of yield; (iii) Can the given loading be sustained by the material if material does not harden?

Mechanics
PhD Preliminary Spring 2017

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1. (10 points) Consider a body Ω that is assembled by gluing together two separate bodies along a flat interface. The normal vector to the interface is given by $\boldsymbol{n} = (1\ 1\ 1)^T$. Assume that Ω is subjected to a homogeneous (Cauchy) stress field

$$\boldsymbol{\sigma} = \beta \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix} \text{ MPa}$$

where β is a non-dimensional load factor. Assume that the interface fails when the normal stress on the interface exceeds 100 MPa. Determine the maximal permissible value of β.

2. (10 points) Consider a small-strain hyperelastic body Ω, free of body forces, with strain energy density $W(\boldsymbol{\varepsilon})$ subjected to dead-loads $\bar{\boldsymbol{t}}$ on $\partial\Omega_t \subset \partial\Omega$ and restrained with zero displacement on $\partial\Omega_u \subset \partial\Omega$, where $\overline{\partial\Omega_t \cap \partial\Omega_u} = \partial\Omega$. Starting from the principal of minimum potential energy, derive the strong form of the governing equilibrium equations.

3. (10 points) A three dimensional viscoelastic body $\Omega = \{\boldsymbol{x} \mid x_1^2 + x_2^2 \leq R^2 \text{ and } |x_3| \leq L/2\}$ has a measured displacement field $\boldsymbol{u} = (\alpha x_1/R)\boldsymbol{e}_1 + (\alpha x_2/R)\boldsymbol{e}_2$, where R and L are given and $\alpha > 0$ is much smaller than R and L. Determine the algebraically maximum and minimum normal strains in the body as well as the maximum shear strain in the body.

Mechanics
PhD Preliminary Fall 2016

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1. (10 points) Consider a linear elastic, isotropic material, under two-dimensional plane strain. Assume that the material is subjected to a body force, $\boldsymbol{b}$, that is described by a known potential, V; i.e. $b_i = -V_{,i}$. Assume further that

$$\begin{aligned} \sigma_{11} &= \phi_{,22} + V \\ \sigma_{22} &= \phi_{,11} + V \\ \sigma_{12} &= -\phi_{,12} \end{aligned}$$

where ϕ is an unknown stress function.

   (a) Show that the equilibrium equations are identically satisfied by this assumption.

   (b) Noting that compatibility requires $\varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} = 0$, find the governing equation for ϕ in terms of V and the elastic constants.

2. (10 points) The general form of an infinitesimal rigid motion in two-dimensions can be described by three scalars A, B, C where A, B describe an infinitesimal translation and C describes an infinitesimal rotation. In Cartesian form the resulting rigid displacement field is given by:

$$\begin{aligned} u_1(x_1, x_2) &= A + Cx_2 \\ u_2(x_1, x_2) &= B - Cx_1 \end{aligned}$$

Find the form of this rigid displacement field in polar coordinates; i.e. determine the corresponding $u_r(r, \theta)$ and $u_\theta(r, \theta)$.

3. (10 points) Starting from the global statement of linear momentum balance

$$\int_{\mathcal{B}} \boldsymbol{b}(\boldsymbol{x})\, dV + \int_{\partial\mathcal{B}} \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n})\, dA = \int_{\mathcal{B}} \rho(\boldsymbol{x})\ddot{\boldsymbol{u}}(\boldsymbol{x})\, dV$$

derive "Newton's 3rd Law" $\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{x}, -\boldsymbol{n})$. *Hint: Employ a pillbox construction.*

UNIVERSITY OF CALIFORNIA AT BERKELEY Department of Civil and Environmental Engineering
SPRING SEMESTER 2016 Structural Engineering, Mechanics and Materials

Name ________________________________

Doctoral Preliminary Examination

Mechanics

Problem 1. (50 points)
Consider the double cantilever beam shown in Figure 1. The beam length is a, width is B and height is h. Assume that the beam is made by linear elastic material, and the Young's modulus is E, and the moment inertia is $I = Bh^3/12$.

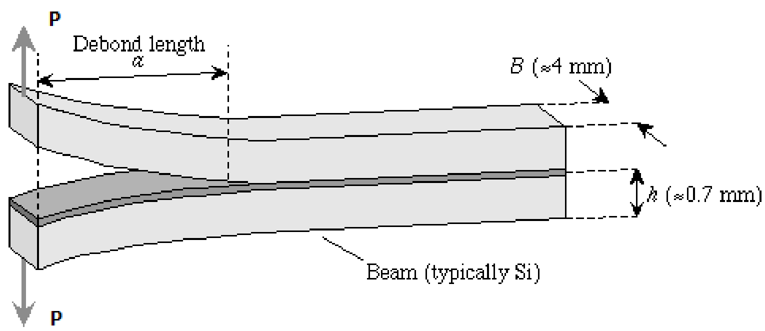

Figure 1: A double cantilever beam

Calculate:
(1) The bending moment $M(x)$ and the total strain energy stored inside the beam U_E (for the sum of the two beams);
(2) The cantilever beam deflection (under the transverse load) at the tip ($EIv''(x) = M(x)$) ?
(3) The total external potential W_L (for the two forces) ?
(4) The total fracture energy release per unit width, i.e.

$$G = -\frac{1}{B}\frac{d\Pi}{da}, \quad \text{where } \Pi = U_E - W_L$$

Hints:
(1) Do not need to calculate numerical values.

(2) Only consider the classical bending part of the strain energy of the beam.

Problem 2.(30 points)
Consider a simply connected continuum media Ω under infinitesimal deformation. The following displacement boundary condition is prescribed,

$$u_i = E_{ij} x_j, \ \ \forall \mathbf{x} \in \partial\Omega,$$

where E_{ij} is a constant symmetric tensor.

Show that

$$\frac{1}{\Omega} \int_\Omega \epsilon_{ij} d\Omega \ , = E_{ij}$$

for the infinitesimal deformation (strain) field ϵ_{ij}.

Problem 3.(20 points)
Derive the weak formulation for the following static linear elastic (small deformation) problem, i.e.

$$\sigma_{ij,j} + f_i = 0, \ \ \forall \mathbf{x} \in V \ , \tag{1}$$

where $\sigma_{ij} = C_{ijk\ell} \epsilon_{k\ell}$ is the Cauchy stress; $\epsilon_{ij} = (1/2)(u_{i,j} + u_{j,i})$ is the strain field, and f_i is the body force. The following displacement boundary condition is being prescribed,

$$u_i = \bar{u}_i, \ \ \forall \mathbf{x} \in \partial V \ , \tag{2}$$

where $\bar{u}_i$ is the prescribed boundary displacement.

Name ________________________________

Doctoral Preliminary Examination (Solid Mechanics)

Problem 1. (40 points)
Consider a single-connected domain Ω. The following boundary condition is prescribed,

$$\sigma_{ij} n_j = t_i = \Sigma_{ij} n_j, \ \ \forall \mathbf{x} \in \partial\Omega$$

where Σ_{ij} is a constant stress tensor.

Assume that the body force is zero, and the equilibrium equation inside the domain has the following form,

$$\sigma_{ij,j} = 0 \ .$$

Show that

$$\frac{1}{\Omega} \int_{\Omega} \sigma_{ij} \epsilon_{ij} d\Omega = \Sigma_{ij} < \epsilon_{ij} >$$

where

$$< \epsilon_{ij} > := \frac{1}{\Omega} \int_{\Omega} \epsilon_{ij} d\Omega \ .$$

Problem 2. (30 points)
Consider the following displacement field,

$$\begin{aligned} u_x &= \frac{1}{2} y^2 + \frac{1}{4} y^4 + xz \\ u_y &= \frac{1}{2} x^2 + \frac{1}{4} x^4 + yz \\ u_z &= xy \end{aligned}$$

1. Find the strain field ?
2. Find the rotation field ?
3. At the point $(1, 1, 0)$, there is a principal strain that has value -2.0. Find the other two principal strains.

Problem 3. (30 points)
Consider a stress tensor in plane stress state,

$$[\sigma_{ij}] = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$$

Find suitable planes, i.e. $\mathbf{n} = (n_1, n_2)^T$, such that $\sigma_n = 0$.

UNIVERSITY OF CALIFORNIA AT BERKELEY
SPRING SEMESTER 2015

Department of Civil and Environmental Engineering
Structural Engineering, Mechanics and Materials

Name ________________________________

Doctoral Preliminary Examination

Mechanics

Problem 1(25 points)
Consider the following plane strain state,

$$\begin{aligned} \epsilon_{xx} &= a_1 y^2 + a_2 y^4 \\ \epsilon_{yy} &= b_1 x^2 + b_2 x^4 \\ \gamma_{xy} &= cxy(x^2 + y^2) \end{aligned}$$

1. Is this strain field a compatible strain field ?
   If it is, are any restrictions required on the constants $a's, b's$ and c ?

2. Find the principal strains at the point $(1, 1)$ with the constant $a_1 = 1$ and $c = 1$.

Problem 2. (50 points)
The stress components at a point P is given as

$$[\sigma_{ij}] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix}$$

(a) Find the traction components on a plane passing through point P, whose outward normal is $\mathbf{n} = (3/5, 0, 4/5)^T$;
(b) Find the normal and shear stresses at the point P on that given plane, and
(c) Find the principal stresses at the point P.

Problem 3. (25 points)

Consider the isotropic linear elasticity stress-strain relation,

$$\sigma_{ij} = \lambda \epsilon_{pp} \delta_{ij} + 2\mu \epsilon_{ij} \ .$$

where λ and μ are Lamé constants.

Consider that an isotropic elastic sphere of radius r centred at the origin undergoes the following uniform dilatation deformation,

$$u_1 = \alpha x_1,\ u_2 = \alpha x_2,\ \text{ and }\ u_3 = \alpha x_3\ ,\ \ \alpha > 0\ .$$

(a) Find the strain components;

(b) Find the stress components.

(c) Derive the bulk modulus of the material.

$$\text{Hint}:\ K := \frac{\sigma_m}{\epsilon_{ii}},\quad \sigma_m = \frac{1}{3}\sigma_{ii}\ .$$